Lecture 39 of Adrian Ocneanu

Notes by the Harvard Group

Lecture notes for 4 December 2017.

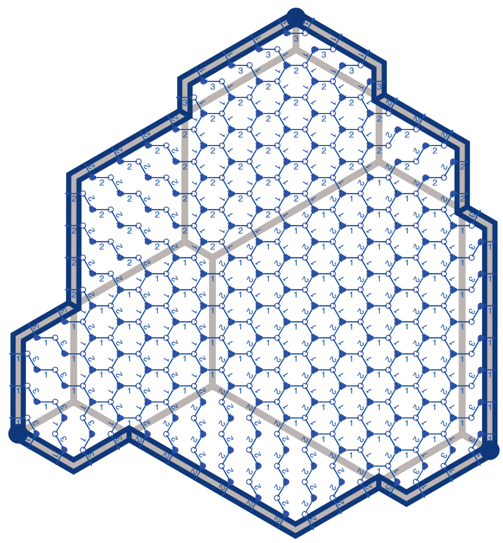


Figure 1: Symmetrizer for $sl(4)$

Consider the standard representation:

$$sl(n) \curvearrowright V = \mathbb{C}^n$$

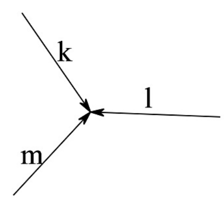
Assume $k + l + m = n$, then you have

$$V^{\wedge k} \otimes V^{\wedge l} \otimes V^{\wedge m} \supseteq V^{\wedge k} \wedge V^{\wedge l} \wedge V^{\wedge m} = V^{\wedge n} \equiv 1$$

it means that if you write

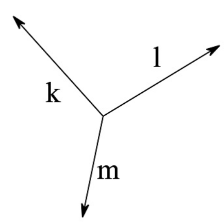
$$^{id}V^{\wedge k}$$

then you have



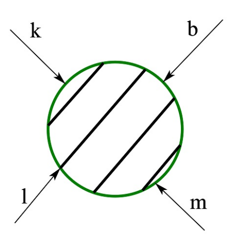
and nothing goes out. So you have the intertwiner of this form; this is the fundamental intertwiner.

You also have its adjoint, when you have k, l, m, going outside:

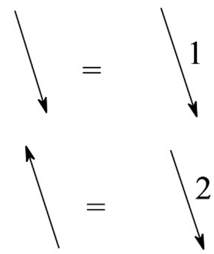


So this is the simply linear conjugate of the previous one. And in this case, if you write out the k's, that corresponds to sum of this form $k' + l' + m' = 2n$, so basically what you want is $k + l + m \equiv 0$ mod n. These are the basic intertwiners.

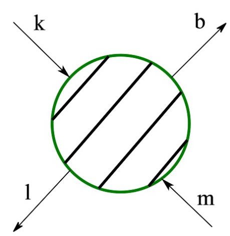
You have some normalizing relation and some very interesting quadratic ones. The quadratic relation for $sl(3)$, given by Kupfelberg (the general case for $sl(n)$ was given by Murakami) is the following: you take a blob like the one below, which has some generators k, l, m, b:



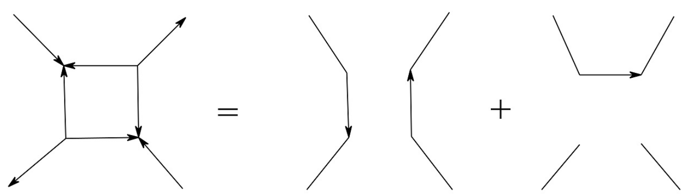
and you compute its dimension, which is very easy because you just tensor here k with l with m with l. So you take the Homs form k tensor l, and you know how to decompose k tensor l, you know how to decompose tensor and b, and the number of Homs will then be computable very easily. So if you write more intertwiners then there are here you get a linear relation. In the $sl(3)$ case, if you have



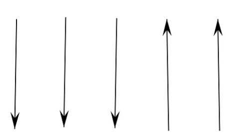
then since two is a conjugate, $V^{\wedge 2} \equiv \overline{V}$. These are called 3 and $\overline{3}$ by physicists. So if you have $sl(3)$ and you take the alternating picture below then you can find very easily that the dimension of



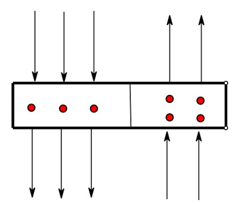
this is 2. So if you are going to write 3 of them, they are going to be linear dependent. In our case, the three of them are given below. All of these are intertwiners, so only 2 of them can be independent,



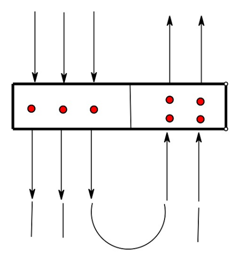
and the third one is the linear combination of the 2. Using this, one more ingredient that you'll have is the following. If you take something like this: this is the identity of $V^{\otimes 3} \otimes (V^{\wedge 2})^{\otimes 2}$, and you can



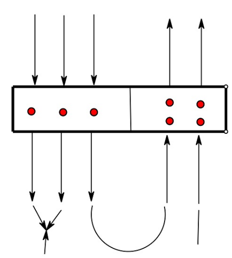
take the irreducible component out of this. This is the projection on a Young diagram. This is just a symbol, it is a projection. But it's calculated by the fact that this is the identity plus lower. The identity has a coefficient 1 in it, and moreover we know from representation theory that the tensor



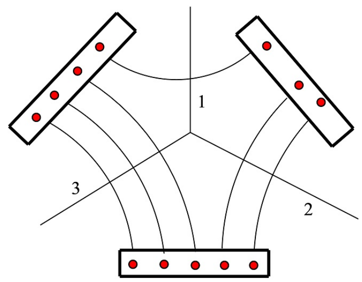
product of two irreducible representation of semisimple Lie groups contains a highest weight, plus some lower things, which you treat as perturbations. This is the highest weight, so in particular if you put here a cap like this at the bottom, you have something on fewer wires, on only 3 wires. This



irreducible does not exist on 3 wires, so it means when you put a cap like thisit makes the pictures zero. There is no map from the reducible to a tensor product of fewer terms. Similarly, if you put here two things like

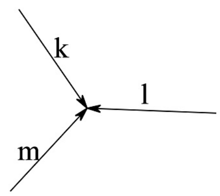


this also makes fewer wires. The representation can not appear there, so again this gives you zero. So if you cap it like this it gives you zero, and this allows you to compute this projection by induction. So these 2 properties give you the projection, and then you take these projections, and you take 3 of projections like this (see video 12:00):



That is a typical intertwiner for $sl(2)$. You can put here something of length 1, 3, and 2. It's a graphical symbol which tells you exactly what things do. 2 of the generators moves right, 3 moves left, and one is contracted. You can really compute with this, because you have a product something like $e_1 \otimes e_1 \otimes e_2$, and something you know like $e_2 \otimes e_1$. That maps into the identity. $V \wedge V \supseteq 1$. So if you have an e_1 and an e_2 they pair, and if you have $e_1 \otimes e_1$, that gives you zero.

Figure 1 shows the symmetrizer for $sl(4)$. It contains precisely pieces like this



for k, l, m. It is well-defined, because the picture is invariant to rotation.

(See video 16:20, for the analysis of Figure 1)

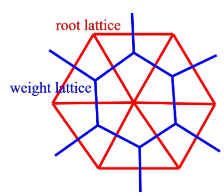
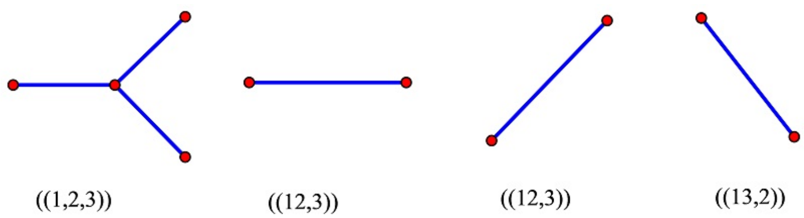


Figure 2: Permutohedra centered around weights

The red ones are the root lattice, and the blue ones are the weight lattice. If you take the von-Neumann neighbourhood of weights, then you get the permutahedron. This shows you that the intertwiner at this place is a computing machine. (see video 23:10, how to compute the determinant, and analysis of Figure 1)

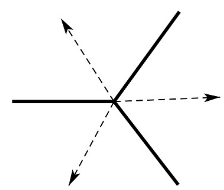
Blades were pieces of the affine paving by permutohedra. The plates will give you something like a rooftop (see video 29:40).



if you have something like

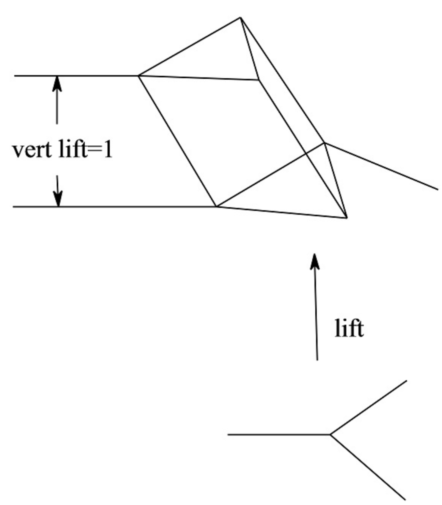
$$\frac{(12)(345)(67)}{00\ \ 111\ \ 22}$$

then you take this vector which has here jumps of 1, and then you divide by the number of lumps, which is 3 here. So the average is 0, so that way you are going to get a vector like this:



Then you use the inner product of each plate, and this gives you something called the potential of a blade. It's linear on each of the regions, because it's an inner product. So what you get is exactly a rooftop, with 3 slopes.

Once we have the *lift*



(our original picture is in the root lattice), we are going to really lift every piece orthogonally,such that for every point the vertical lift is 1. (see video 38:15, lift a point of lattice, and other picture of lift) At the top of a lift is an intertwiner.

Main theorem Linear combinations of codimension 1 blades are exactly the functions from the span of the roots to $\mathbb{R}$, which are continues, and linear except on the special hyperplanes $\{x_S \in \mathbb{Z}\}$.

Saying that the maps are linear except on the special hyperplanes $\{x_S \in \mathbb{Z}\}$ is the same as saying that they are linear on shards.

The rooftop-like functions coincide on the intersection, so there will be a bent in the direction normal to the intersection. (see video 44:00) This bend satisfies the conservation relation because if you take 6, intersection of, say, 3 hyperplanes, the multiplicity of edges on one side is equal to that on the other side.

What you show is that if this conservation relation satisfied, then they are linear combination of generators.

(See video 45:40, for a geode, Young diagram bending a wire, curvature.) Once we reduce this the intertwiners to curvature, a computation of that formal intertwiner is interpreted as the inner underlying; in a way it's matter itself. And as you can see with this model the matter lives in the folds of the underlying space.

(See video 51:30 for animations of: bending and lift, curvature and honeycomb of 1 dimension higher, interaction between interactors.)