Lecture 40 of Adrian Ocneanu

Notes by the Harvard Group

Lecture notes for 6 December 2017.

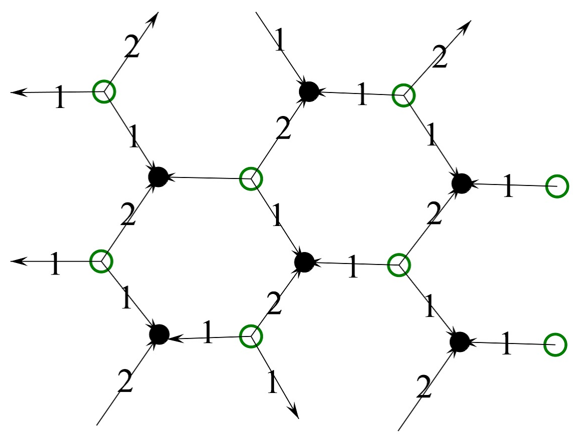


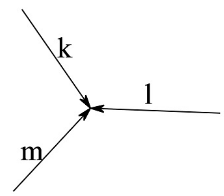
Figure 1: The picture Adrian showed at the beginning

What you see here (Figure **??**) is exactly those elementary bits of intertwiners. As we saw, for $k+l+m=n$

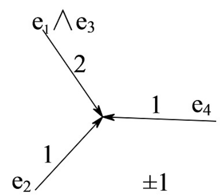
$$sl(n) \curvearrowright V = \mathbb{C}^n$$

$$V^k \otimes V^l \otimes V^m \to V^{\wedge k+l+m} = \det = 1$$

What we have, graphicly is three arrows, and $sl(n)$ acts on V which is $\mathbb{C}^n$, the standard space, and



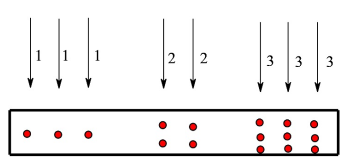
this is the standard intertwiner, right? In the figure above nothing goes out, so this is invariant, and the det is 1. So this simply computes determinants. On $sl(4)$ one can put, for example, and this



gives you 1 up to a minus sign. In other case the reult can be 0 If you have a general representation (not the generators) you have some symmetrizer, so this things on the edge are the projections onto the irreducible representation (see 4:50). You have a tensor products of the generators, and the projection onto the irreducible (representation). These are the 3 projections which are on the side. This was studied by Hermann Weyl. If you take

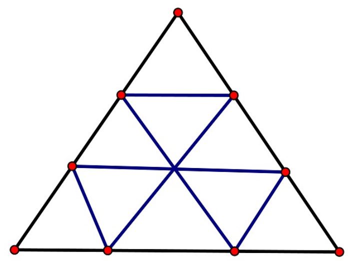
$$(V^{\wedge 1})^{\otimes a} \otimes (V^{\wedge 2})^{\otimes b} \otimes (V^{\wedge 3})^{\otimes c}$$

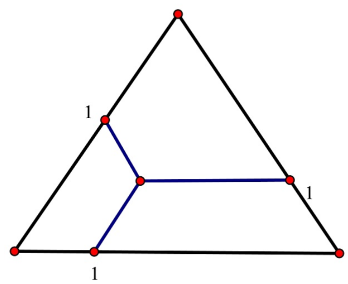
you can write this as arrows, then put a symmetrizer, and you get the Young diagram: So you get



the fundamental representation. If you start with some representation you end up in some regions, and these regions are each of certain kind. This is what I called phases. So you have these regions, they are separated like this.

The main idea is the following. We have a triangulation, which will give special hyperplanes, and then at every point we get a lift. For every edge we have some multiplicity:



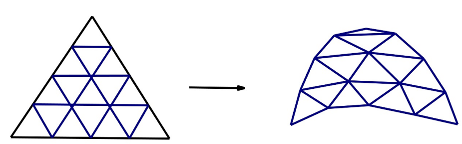


An intertwiner is of this form:
We call this a blade. This is going to produce a lift, and the lift is exactly like a rooftop. So blades like this produce some bend. If you want to compute the bending, we use the definition of Gaussian curvature (see video, 12:50).

This is what exactly what we have here (see video 13:35). In the 2-dimensional case we bend the triangles by adding a height, and the height is given by some potential.

This must be clear.

Once we curve the space



there is going to Gaussian curvature at each point. And here we arise it. We arise a normal at every point. For each of them we give one of the corners. Then we also put the missing parts, and what we get is exactly the picture thta we had before. The type of the intertwiner is exactly the coordinate of the corresponding point in the base.

The length of these edges is a number (see video 16:54on the edge; here you have 6 blades, so the 6 blades became a length 6. They move the part exactly the multiplicity of the respective blade. The conservation relation that we have before tells us that the length of the two edges of the hexagon at the top is the same as the length of the two edges at the bottom. The theorem is that any intertwiner is a linear combination of these. In any intertwiner is described by a honey comb, we separate the regions into intertwiners of the same kind: in any of these regions the intertwiners are exactly of the

4 0 0

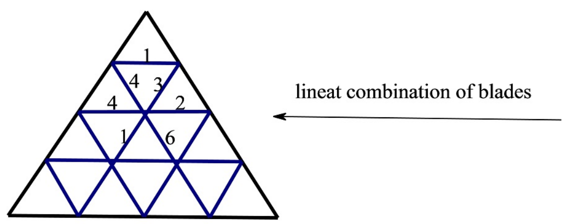
2 1 1

0 4 0 0 0 4

same kind. The honeycomb are exactly the basis of roots of sl(3). In fact, the number of honeycombs is exactly the number of the intertwiners. So they reduce the theory of 1960's. It took almost 20 years to proof

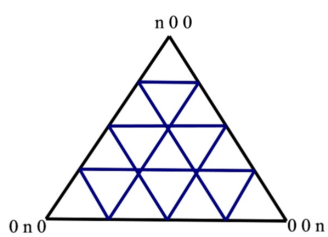
Question: how is this related to the picture you mention for curvatures?

Answer: Let's look at the picture:



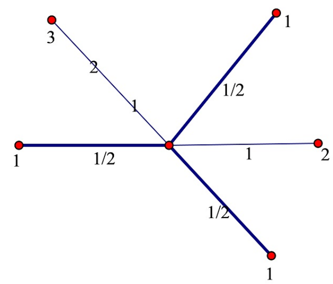
Question: what is the relation to the curvature?

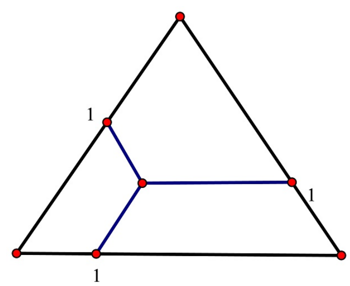
Answer: the curvature is as follows. You take the simplex, with coordinates:



each vertex becomes an eye of the honeycomb. (see video 31:00)

This gives exactly the shape. For every blade there is a *height*, then you add heights of every blade, so the height is a linear function. If you take a blade like this





it gives you *height function*. The theorem is that, if you have some multiplicities which satisfy conservation relations, then they are the linear combinations of these standard blades. The blades are automatically satisfy the conservation relation.

Question: I don't see the statements of the theorems.

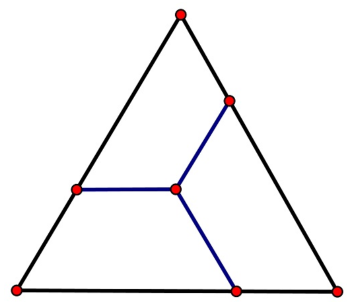
Answer: These are the statements. The multiplicity automatically satisfy the conservation relations. Each blade has a well-defined height, exactly through this formula. There is a vector for each region, which in the nondegenerate case is exactly the Weyl vector ρ. Each blade is a simple root.

The Gaussian curvature is exactly the area. If given an area, if you know a Gaussian curvature, you could exactly give a certain class of intertwiners. The number of honeycombs is equal to the number of intertwiners. That is what we know in the case of Gaussian curvature. So heights give numbers at every point. What's known is that the length of the edge of the honeycomb is equal to the sum of these minus the sum of these (51:20). So they don't see the geode at all. That's why they could not be generated to higher dimensions. To this point we only need condiitons of integrality and convexity.

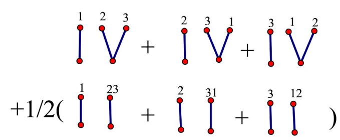
In the 2-dimensional case the height is known. It is known that the intertwiners are also labeled by "hives". What is not known is the geometrical connection between heights and honeycombs, which is the geode that is showing here (52:15). And what is also not known is that these honeycombs give you actually intertwiners by filling each eye of the honeycomb with particular intertwiner exactly this.

This is what the poster is about: the honeycombs describe exactly intertwiners, and so any intertwiner can be reduced to a linear combination of these.

Consider the standard generator



which is the anti symmetrized sum of trees

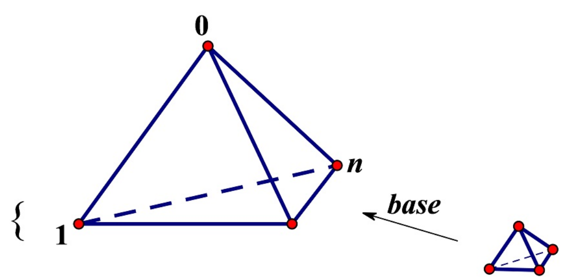


Question: what is the relation to Gaussian curvature?

Answer: Well, this generator will expand, and it gives you triangles (62:50), so these things give you Gaussian curvature.

Having seen the intertwiner, how do we have higher matrices act? Today we will define how matrix units act on intertwiners.

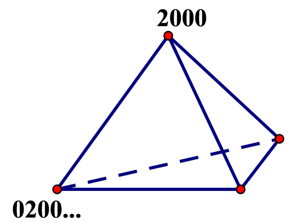
We take $sl(2)/sl(n)$. This is a simplex. The length of the edges is 2. In that case, what we have



in the base (the small one), are blades which are of the following type: they separate the coordinates into 2 (this particular one is $x_{12} = x$), so the coordinates of the simplex are something like this:

This is $x_S = 1$, where S is a subset of the coordinates $\{1, 2, ..., n\}$. So every blade here separate the coordinates into two. You see here that you have the coordinates separated into 1,2,3 and 4, and the blade is the square (69:34). That is actually the motivation for these special hyperplanes.

What you have in the base is the intertwiner, and out of this intertwiner you grow blades (this is exactly what we did in Gelfand-Tsetlin). The edge on the base can grow either in a square or in a triangle. Here you have the bottom $12, 3$, the blade that separates the vertex 12 and 3, and you



can go either in a triangle, or in a square in this other direction. It goes either in 012,3, which is a triangle, or to 12,03, which is a square. The way these grow is exactly by adding 0 on one side or the other side. This works for any number of coordinates. Matrix units will tell us how to bend these leaves.

The idea is the following: the basis is given by intertwinters. Out of these you grow vegetables, which you bend with the higher matrices. If you only want invariant vectors, you get Wigner $3j$'s; but this is too restrictive, and higher representations allow us to do whatever we want.

By the way, there is a formula I intend to do on Friday at the beginning,