The one dimensional TQFT

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The one dimensional theory is simple, but shows in full detail the way in which the dictionary between space and algebraic computations works.

A one dimensional manifold with boundary is an oriented interval with two endpoints, which we orient with a sign $+$ and respectively $-$.

The pieces of the boundary are mapped to Hilbert spaces, here finite dimensional, for the purpose of gluing with other such pieces by taking inner products. QFT is tensorial, so the spaces here are V (for the endpoint with $+$ orientation) and a $\overline{V}$ (for the endpoint with the $-$ orientation), where the conjugate space $\overline{V}$ is a space with vectors $\overline{v}$ for $v \in V$, having scalar multiplication and inner products conjugated.

The interior of the interval contracts the endpoints, and thus algebraically maps $V \otimes \overline{V} \to \mathbb{C}$. Remark that in this view, the two endpoints are **entries**; there are no exits. One inputs two vectors in them and one gets a number in $\mathbb{C}$, where $\mathbb{C}$ corresponds to the empty set as exit. Recall that QFT is exponential, so the empty set maps to the scalars $\mathbb{C}$.

If we use the $-$ boundary as entry and the $+$ boundary as exit, we get a map $P : V \to V$.

The flipped (reversely oriented) gives the adjoint map P^*. Gluing an exit to an entry gives $P^2 : V \to V$. As our main axiom in QFT is that a manifold does not depend of its subdivision, we have $P = P^2 = P^*$ so P is an orthogonal projection. Think of the interior of the interval, physically the **matter** which fills it, as a kind of computation machine, which in this simple case acts by a projection on the vectors which pass through.

Entries and exits

Let us change the role of the endpoints. Any part of the boundary can be used as either entry or exit. Graphically a vector v near a boundary part means that it is an entry and we input v.

If we input a vector $v \in V$ at the $-$ end, we get at the other end Pv.

If we input vectors $v \in V$ at the $-$ end and $\overline{w} \in \overline{V}$ at the $+$ end we have no exit and get the scalar $\langle Pv, w \rangle$.

If we input nothing (which in fact is $1 \in \mathbb{C}$), we get at the two exits

$$\sum_{v \in V} Pv \otimes \overline{v} \in V \otimes \overline{V}$$

where by abuse of notation $\sum_{v \in V}$ is taken to mean the sum over an orthonormal basis of V. The result does not depend of the choice of basis.

Manifold invariants

We can make a closed manifold, a circle obtained by gluing the + and − endpoints. That means we sum over all v in an orthonormal basis of V the numbers $\langle Pv, v\rangle$. The scalar obtained is the trace of P, which is its rank.

We now consider **types**, which are labels in a set L for boundary pieces. Labels need to match when pieces are glued, else the gluing makes no sense, or is defined as 0.

That gives operators $P_{ij} : V_j \to V_i$ for $i, j \in L$, with $P_ijP_{jk} = P_{ik}$ and $P^*_{ij} = P_{ji}$.

Note that in this case $\mathrm{tr}P_{jj} = \mathrm{tr}(P_{ji}Pij = \mathrm{tr}(P_{ij}Pji = \mathrm{tr}P_{ii}$. Topologically this translates into the fact that the number corresponding to the closed 1 manifold, the circle, triangulated with one point does not depend on the type, i or j, of that inner point.

Solving the theory

Let us convene that $1 \in L$ but $0 \notin L$.We choose an orthonormal basis for V_1. The columns of the matrix of the projection P_{11} are the images of the basis vectors. On the algebraic side, we do the Gram-Schmidt orthonormalization and we obtain a basis of the image of P_{11}. We call this new vector space V_0. We make the basis vectors the columns of a new matrix $P_{10} : V_0 \to V_1$, and we call $P_{01} = P^*_{10}$.

The algebraic construction gives an orthonormal basis, which translates into $P_{10}P_{01} = P11$ and P_{00} defined by $P_{00} = P_{01}P10$ the identity of the new space V_0, that is, P_{10} is an isometry inserting V_0 into V_1.

How should the identity be represented graphically, or rather how should it act as a manifold? Since the interior of the interval corresponding to P_{00} does nothing to the vectors which move through it, the interval representing the identity should **contract** to a point. We still need to assign that point two orientations + and − on two opposite sides.

We now let for any type $i \in .L$ $P_{i0} = P_{i1}P_{10}$ and $P_{0i} = P_{01}P_{1i}$. We have

$$P_{0i}Pij = P_{01}P_{1i}Pij = P_{01}P_{1j} = P_{0j}$$

and

$$P_{i0}P0j = P_{i1}P_{10}P_{01}P_{1j} = P_{i1}P_{11}P_{1j} = P_{ij}.$$

The computations above show precisely how the new label 0 functions. It was defined by graphically cutting P_{11} in two, $P_{11} = P_{10}P_{01}$ and in any expression 0 is either on the boundary, or if it is inside it is surrounded by $P_{10}P_{01}$ and evaluates to P_{11}, back in the old pieces.

The algebraic Gram-Schmidt computation has **enriched** or **augmented** our system with a **new type** 0. The new type **solves** our system, in the sense that any P_{ij} splits in two, with 0 in the middle, as $P_{ij} = P_{i0}P_{0j}$ and since $P_{00} = \mathrm{id}_{\mathrm{V}_0}$ and contracts to a point, the product $P_{ij}P_{jk}$ is defined by contracting the inner two halves to a point.

Algebraic computations, here the Gram-Schmidt algorithm, translate via the QFT dictionary into new objects which match the old ones. The theory is solved in the sense that computations are reduced to inner products, graphically gluings.

One dimensional TQFT and the Dirac Bra-Ket The idea that an operator A is best drawn, or even written, with two endpoints $|A|$ which can be used as both entrances and exits is in fact due to Dirac. In his bra-ket notation, with both endpoints used as entries filled with vectors we get the number $\langle v|A|w\rangle$. When both endpoints are used as exits, we get A as a tensor

$$A = \sum_{v,w} \langle v|A|w\rangle \ |v\rangle \otimes \langle w|$$

The gluing of two endpoints is done by summing over an orthonormal basis of the corresponding boundary vector space

$$|AB| = |A|B| = \sum_{v} |A|v\rangle\langle v|B|$$

An operator which can be subdivided and reversed, like empty space, is $|P| = |P^{*}| = |P|P|$, that is a projection, and a projection is solved by Gram-Schmidt as

$$|P| = \sum_{w \in \mathrm{P(V)}} |w\rangle\langle w|$$

The identity contracts topologically to a point

$$\langle a|1|b\rangle = \sum_{v} \langle a|v\rangle\langle v|b\rangle = \langle a|b\rangle \mathrm{so} |1| = \sum_{\mathrm{v}} |\mathrm{v}\rangle\langle \mathrm{v}| = |.$$

If we have several types of vacuum, we get instead of a projection P operators which we can write as $P_{ij} = |_{i}P|_{j}$ for which

$$|_{i}P|_{j}P|_{k} = |_{i}P|_{k}$$

Gluing the endpoints of an operator together, which was typographically impossible in the time of Dirac, is done with a vector basis as

$$\sum_v \langle v|A|v\rangle = Tr(A)$$

So the one dimensional QFT gives precisely the translation of the algebraic computation with vectors and operators into the very successful one dimensional notation of Dirac.

Thus the one dimensional TQFT describes a one dimensional kind of matter which is **homogeneous, isotropic, infinitely subdivisible, and lets only a linear subspace of vectors pass through**. Which is precisely the physical description of an orthogonal projection.

Conclusions The (pictureless) discussion above shows how space, here a one dimensional interval, functions as a computation machine, on vector data entering and exiting parts of its boundary. Which computation? We need to specify it, and it should satisfy the mathematical structure given by the topological gluing properties. Physically, that is the **matter** that we put into our QFT model.

The model suggests algebraic computations, using the Gram-Schmidt algorithm, which result in an enrichment or augmentation of the system with a new type 0 for the new vector space V_0 and all the new pieces P_{0i} and P_{i0} which insert it in the old system.

Further developments In 2 dimensions the filling in a triangle with the 3 edges viewed as 2 entries and 1 exit gives a binary bilinear operation. Cutting a square in turn by its two diagonals shows that the operation has the structure of an associative algebra with a trace, a Hilbert algebra. There appear also weights in the gluing process. Solving the theory in this case means block diagonalizing the algebra, a computationally nontrivial operation for which we provide an efficient algorithm. That algorithm includes, as a particular case, the explicit computation of the irreducible representations of a finite group, obtained by block diagonalization of its convolution algebra.

In 3 dimensions the algebraic structure given by 3d space on the vector space of a rhombus 2d surface is a weak Hopf algebra, a structure which includes among others groups, groupoids, group duals. All the algebraic properties of this structure follow from properties of 3d space. No solution, in the sense of reduction of such a structure to combinatorics and inner products, is known. However, the one and two dimensional theories embed, in many different ways, into the three dimensional theory and their solutions give new labels for circles, trinions and the thickening of Feynman diagram by strings. Algebraically this constructs the Drinfeld double, coming from the topology of a wormhole. Among others this gives a combinatorial classification of all the braidings of a fusion system.

Thus what seems to be in the one dimensional case an overly detailed study of intervals introduces techniques and dictionaries which are nontrivial and very powerful in higher dimensions.

In this course, such enrichments lead to the construction and classification of quantum subgroups of the quantum groups introduced by Drinfeld and Jimbo, starting from number theoretic modular invariants. These serve further as higher Dynkin type diagrams, crystallographic type data from which higher representation theory is built.

This is further viewed as data, or matter, which fills space-time tensorially, that is with QFT type properties, in a physical number of dimensions. It seems at first surprising that such higher representations are governed by discrete Riemannian curvature, though maybe less so since the constructions are fundamentally based on the structure of space.