Hydrodynamics and Corrections to Random Matrix Universality

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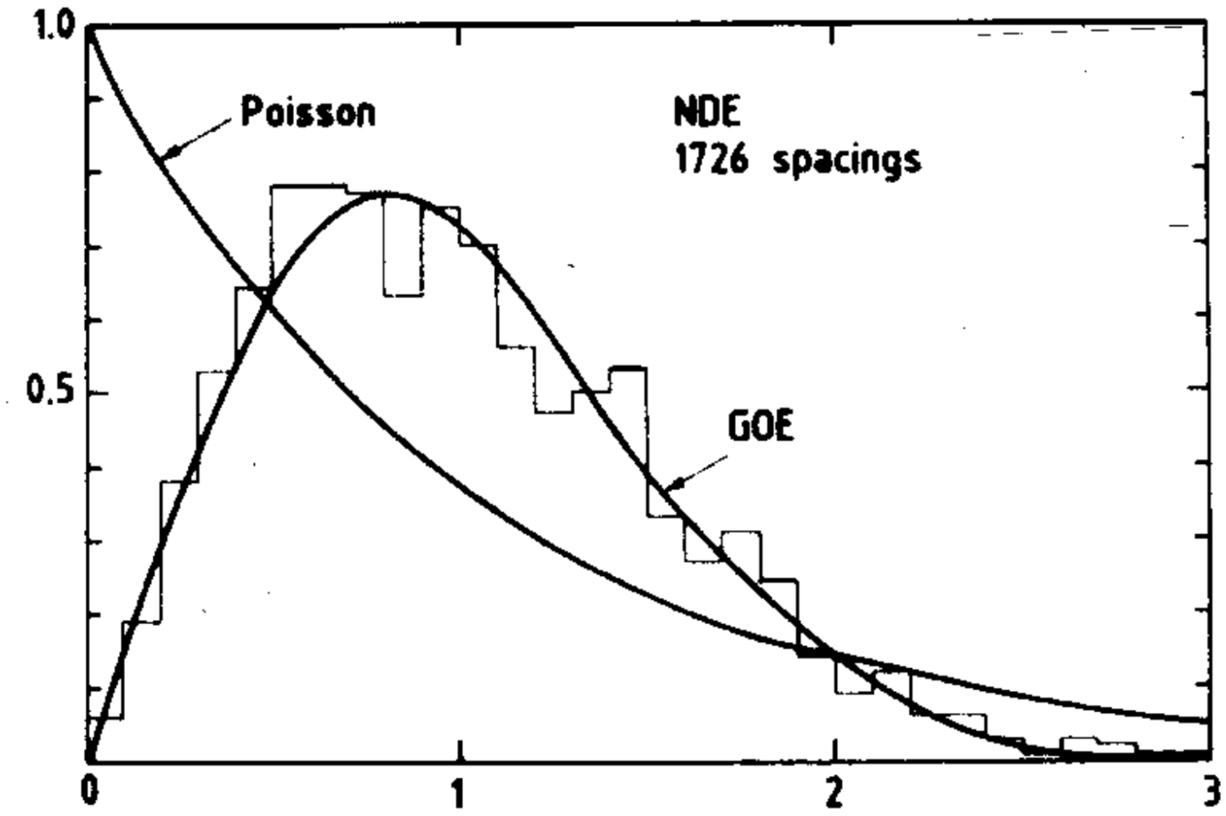
It from Qubit
Simons Collaboration on
Quantum Fields, Gravity and Information

Random matrix theory (RMT) in physics

- Many complex quantum systems have an "unstructured" energy spectrum, especially far away from the ground state
- Wigner's idea: we can model the spectrum of such "unstructured" systems using random* Hermitian matrices [Wigner, Dyson, Mehta, …]

Level spacings of different nuclei with the same spin/parity:



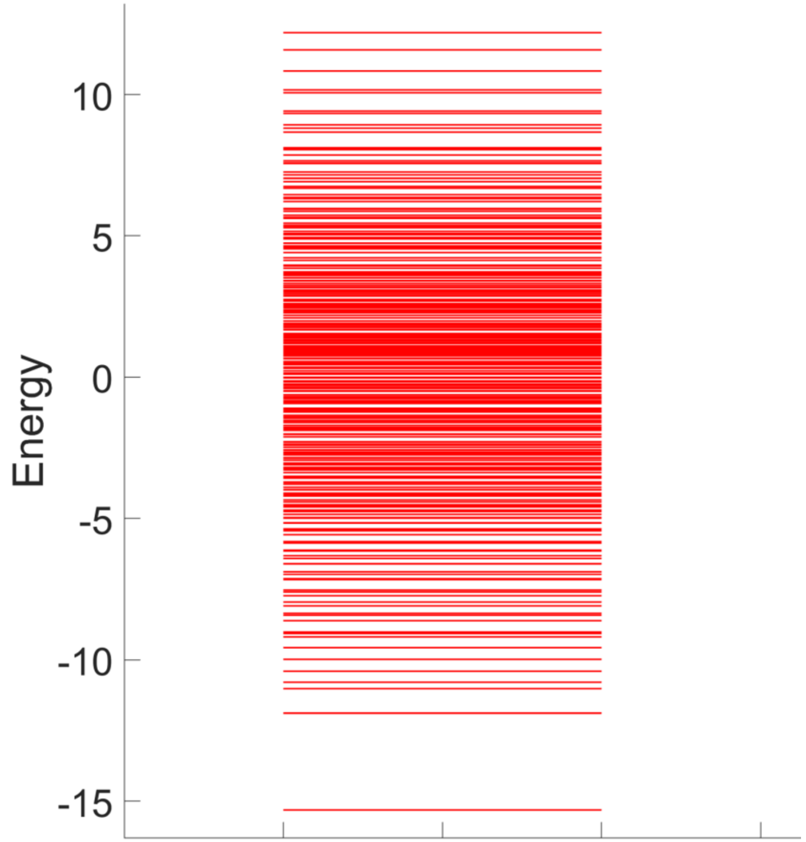
[Bohigas-Haq-Pandey]

*Many types of random matrices, but we'll consider primarily Gaussian ensembles

Nuclei, spin chains, …

$$H = -J \sum_r \sigma_r^z \sigma_{r+1}^z - h_x \sum_r \sigma_r^x - h_z \sum_r \sigma_r^z$$

n=8 spins, 256 energy levels



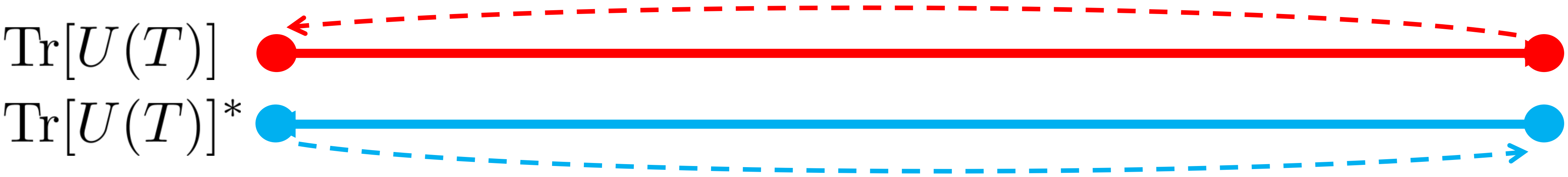
Ensemble of Hamiltonians with random fields:

$$\Delta H = \sum_r \delta h_r \sigma_r^z$$

[chaos-RMT conjecture: Bohigas-Giannoni-Schmit]

This talk – what are the corrections to "pure" RMT in realistic systems?

- *Hydrodynamic theory of the connected spectral form factor*, 2012.01436, w/ Mike Winer (and ongoing work)
- Spectral form factor: $\mathrm{SFF}(T) = \langle |\mathrm{Tr}[U(T)]|^2 \rangle_{\mathrm{disorder}}$.



Chaos → random matrix behavior at "intermediate" time: $\mathrm{SFF}(T) \propto T$

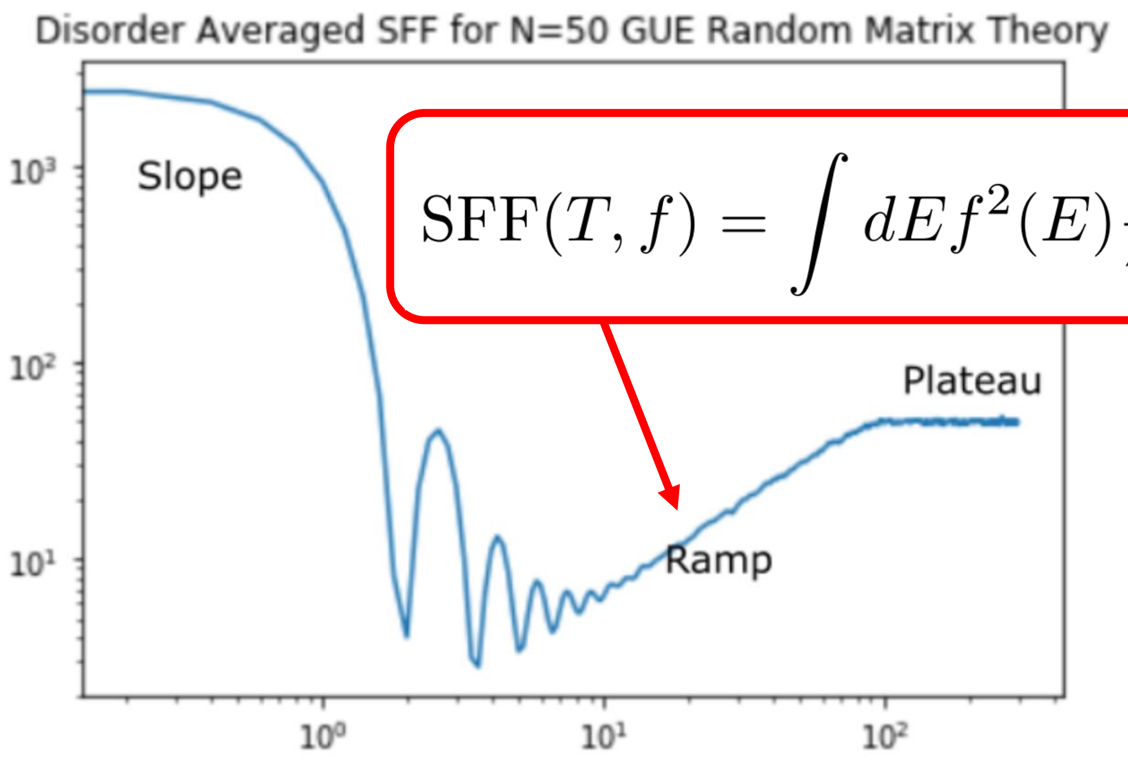
Question: under what conditions is this random matrix behavior realized?

RMT and SFF

$$dP \propto \prod_{ij} dH_{ij} \exp\left(-\mathrm{Tr}[V(H)]\right)$$

$$dP = \frac{1}{\mathcal{Z}} \prod_{i<j} |E_i - E_j|^{\mathfrak{b}} \prod_i e^{-V(E_i)}$$

data: Dyson index and potential



$$\mathrm{SFF}(T, f) = \int dE\, f^2(E) \frac{T}{\pi \mathfrak{b}}$$

$$\mathrm{SFF}(T, f) = \overline{|\mathrm{Tr}[f(H) e^{iHT}]|^2} = \overline{\sum_{i,j} f(E_i) f(E_j) e^{i(E_i - E_j)T}}$$

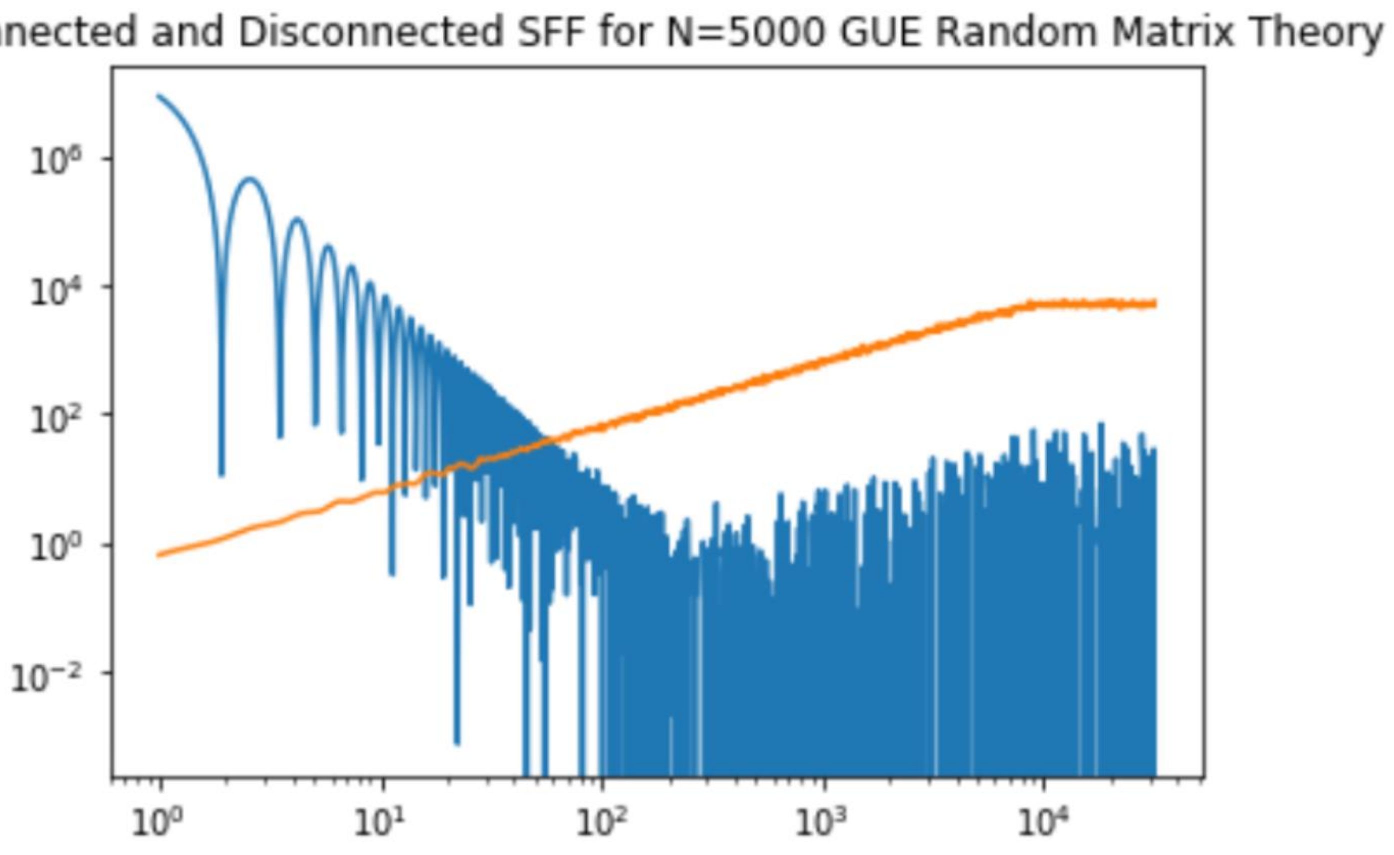
f = filter function

Thouless time

$$Z(T, f) = \mathrm{Tr}[f(H)e^{-iHT}]$$

$$\mathrm{SFF}_{\mathrm{disc}} = |\overline{Z(T,f)}|^2$$

$$\mathrm{SFF}_{\mathrm{conn}} = \mathrm{SFF} - \mathrm{SFF}_{\mathrm{disc}}$$



- In a particular chaotic local Hamiltonian system, random matrix theory will not be accurate at “early” time
- Thouless time = time required to be “close” to the pure random matrix result in the connected SFF, typically at least log(system size)

[early work in many-body context: Chan et al. ’18, Gharibyan et al. ’18, Friedman et al. ‘19]

Plan

We seek an effective theory that predicts random matrix behavior and enables us to compute corrections to that behavior due to structure

- Background: observables of interest, random matrix ensembles
- Example of diffusion: conservation laws and slow modes, effects on the spectrum
- Effective theory: fluctuating hydrodynamics, effective theory of spectral form factor
- Elaborations: sound, glasses, late time

Our work in context

- RMT after diffusive equilibrium: Altshuler-Shklovskii ’86, Argaman-Imry-Smilansky ‘93, reviews: D'Alessio-Kafri-Polkovnikov-Rigol, …
- Many-body RMT onset: Schiulaz-Torres-Herrera-Santos, Gharibyan-Hanada-Shenker-Tezuka, Friedman-Chan-De Luca-Chalker, …
- Many-body analytic results: Bertini-Kos-Prosen, Dubertrand-Muller, Chan-De Luca-Chalker, Saad-Shenker-Stanford, Garcia-Garcia-Verbaarschot, Altland-Sonner, …
- Fluctuating hydrodynamics: Dubovsky-Hui-Nicolis-Son, Grozdanov-Polonyi, Haehl-Loganayagam-Rangamani, Crossley-Glorioso-Liu, Jensen-Pinkani-Fokeeva-Yarom, Chen-Lin-Delacretaz-Hartnoll, …

Example: structure from locality

★ Work with Mike Winer (UMD) [Winer-S 2012.01436] ★

Energy diffusion

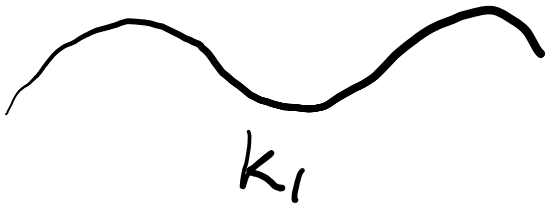
$$\partial_t \epsilon = D\nabla^2\epsilon + \xi$$

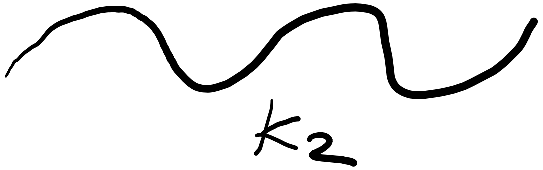
diffusion + "noise" with appropriate correlations

- Imagine breaking all other symmetries: all that remains is energy diffusion → minimal slow dynamics in a local Hamiltonian system
- At time T, there are an extensive number of almost conserved modes:

$$k_T \sim \frac{1}{\sqrt{DT}} \qquad N_T \sim \sum_k \theta(k_T - |k|) \sim V \int \frac{d^d k}{(2\pi)^d} = \frac{V S_d}{(2\pi)^d}\frac{k_T^d}{d}$$

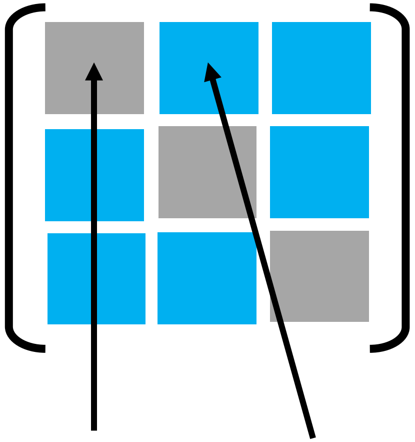
- If each sector is random matrix like, then the SFF should correspond to a sum of many almost-independent ramps → sectors are labelled by amplitudes of nearly-conserved energy fluctuations

$\epsilon(x,t) \simeq \epsilon_{k_1}(t)$ 

$+\ \epsilon_{k_2}(t)$ 

$+ \dots$

Model: nearly-block Hamiltonians



- Decoupled sectors ($\alpha = 1, \cdots, \Omega_0$) + transitions: $H = H_0 + V$
- Decoupled limit, no level repulsion:

$$\mathrm{SFF} \sim T \int \frac{dE}{2\pi} f^2(E) \times (\#\ \text{of sectors at energy}\ E)$$

- Including transitions from V (Fermi's golden rule):

$$\mathrm{SFF}(T, f) = \sum_{\alpha} f(E_\alpha)^2 p_{\alpha\to\alpha}(T)\mathrm{SFF}_\alpha(T)$$

[Winer-S]

Linear diffusion

$$p(\epsilon_{k,\mathrm{final}}, T) = \frac{\exp\left(-\frac{(\epsilon_{k,\mathrm{final}} - e^{-\gamma_k T}\epsilon_k)^2}{2\sigma^2(T)}\right)}{\sqrt{2\pi\sigma^2(T)}}$$

$$\partial_t \epsilon = D\nabla^2 \epsilon + \xi$$

$$\int d\epsilon_k p(\epsilon_{k,\mathrm{final}} = \epsilon_k, T) = \frac{1}{1 - e^{-\gamma_k T}}$$

$$\sum_\alpha p_{\alpha\to\alpha}(T) = \prod_k \frac{1}{1 - e^{-Dk^2 T}} = \exp\left(V\left(\frac{1}{4\pi DT}\right)^{d/2} \zeta(1 + d/2)\right)$$

exclude zero mode, quasi-continuous wavevector regime

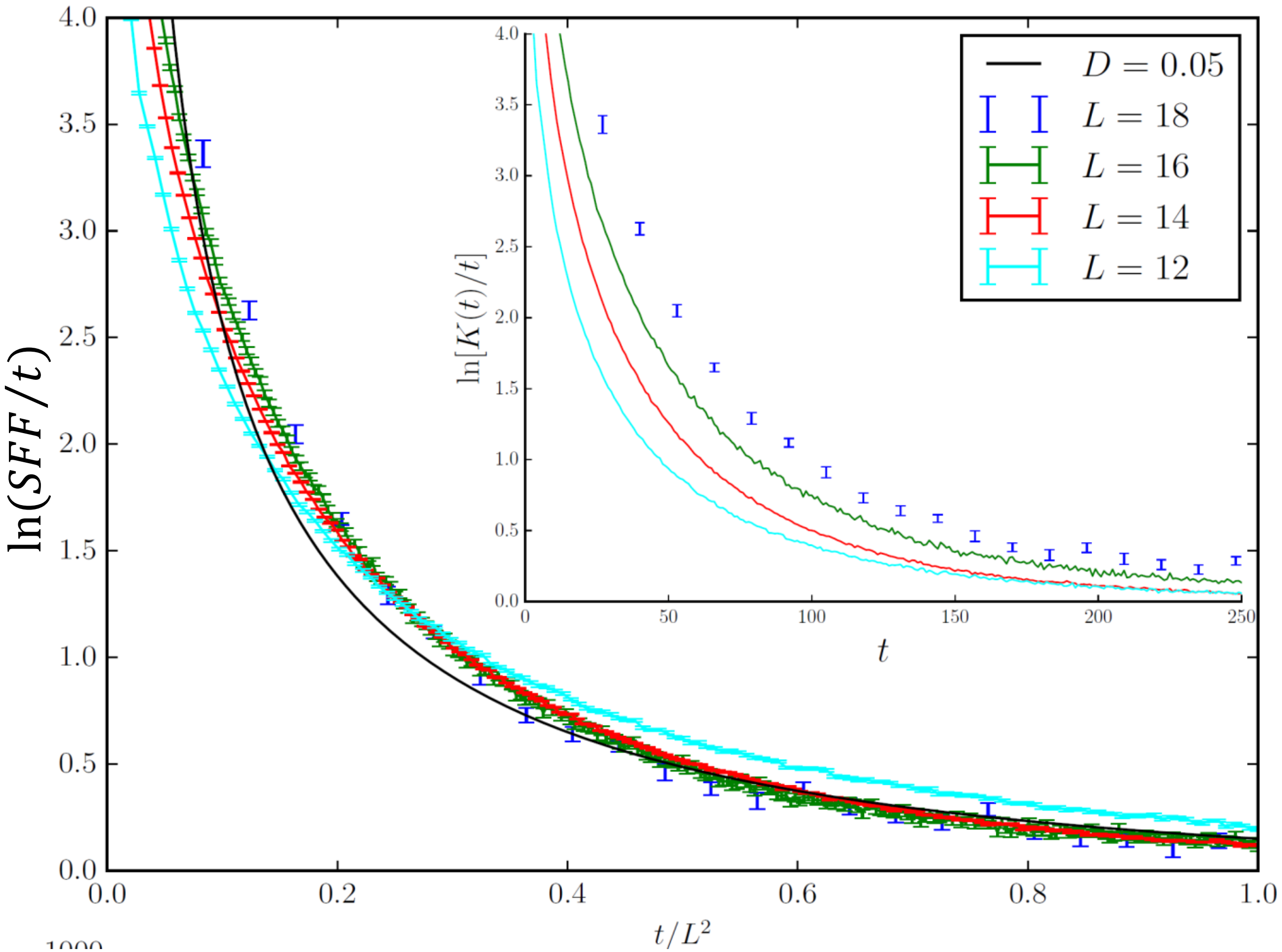
[Winer-S, large-q d=1 Floquet model Friedman et al. ‘19]

$$T = t_{\mathrm{Th}} = \frac{L^2 \log \frac{1}{\epsilon}}{(2\pi)^2 D} \longrightarrow \sum_\alpha p_{\alpha\to\alpha}(T) = 1 + 2d\epsilon + O(\epsilon^2)$$

periodic box

Comparison with numerical data

- Consistent with numerical data from [Friedman et al.], which derives the previous formula (in the context of U(1) conservation) in d=1 with large onsite dimension
- We show that it arises generally from linearized diffusion; and we can compute corrections

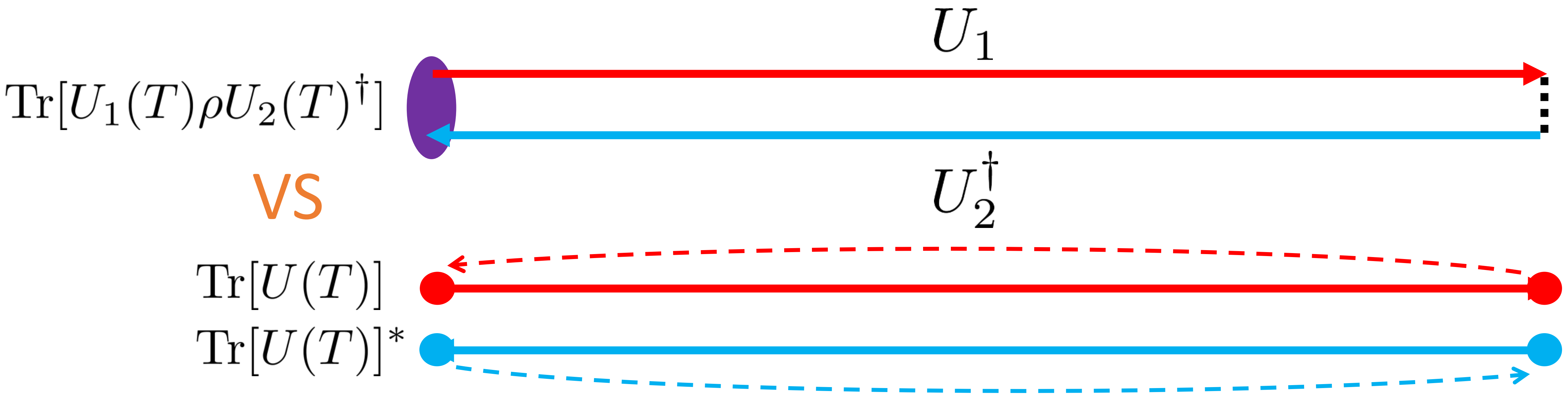


[data from Friedman et al. 1906.07736]

Effective theory

★ Work with Mike Winer (UMD) [Winer-S 2012.01436] ★

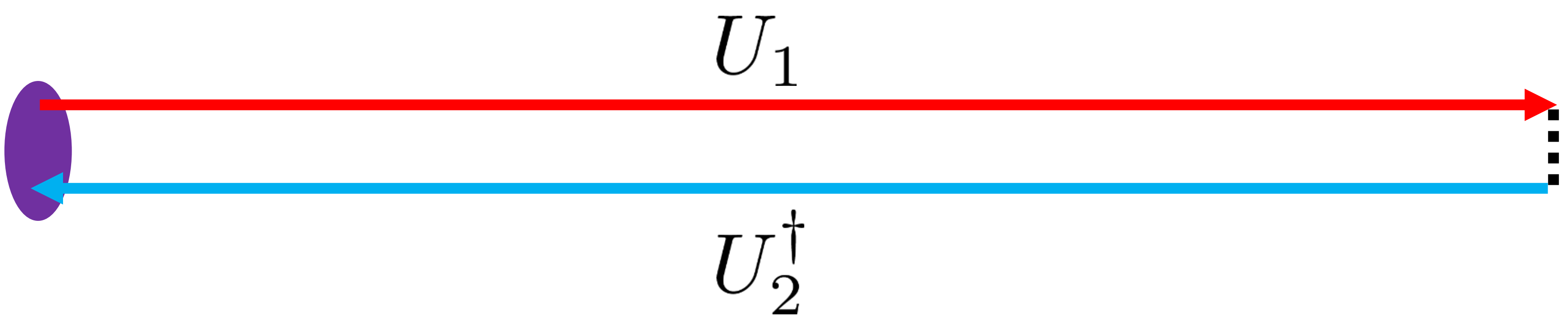
Intuition



SFF effective theory should be related to an effective theory on a Schwinger-Keldysh contour → hydro!

Closed time path (CTP) formalism

$\mathrm{Tr}[U_1(T)\rho U_2(T)^\dagger]$

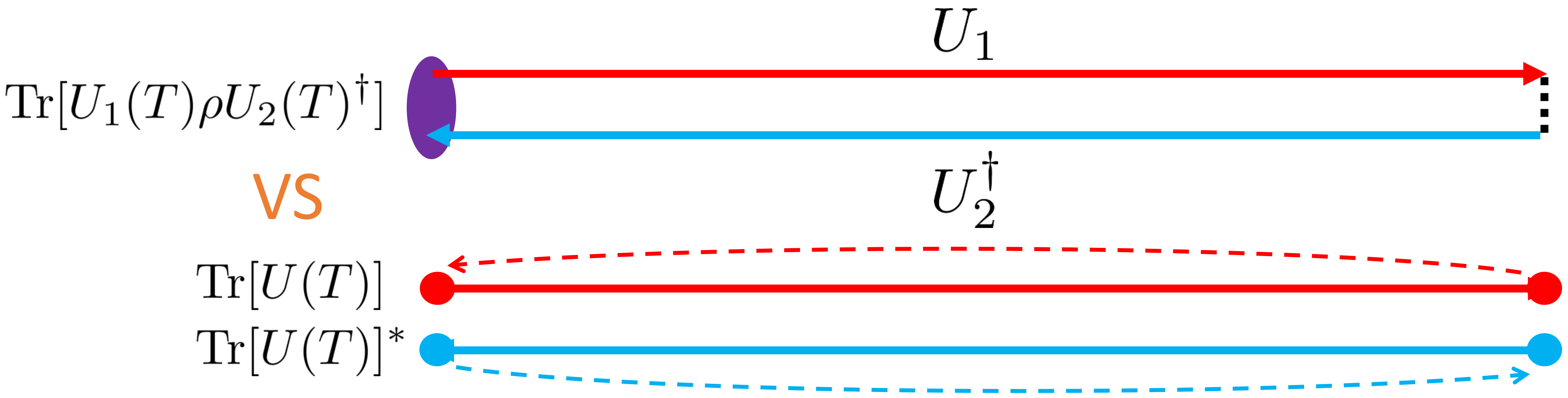


- Symmetric (classical, r-type) and antisymmetric (quantum, a-type) variables, powerful set of rules that govern allowed effective actions
- Let's focus on energy diffusion as a simple example $\epsilon = c\beta^{-1}\partial_t \phi_r$

$$L = -\phi_a(\partial_t - D\Delta)\epsilon + i\beta^{-2}\kappa(\nabla\phi_a)^2$$ [Glorioso-Liu, Chen-Lin et al.]

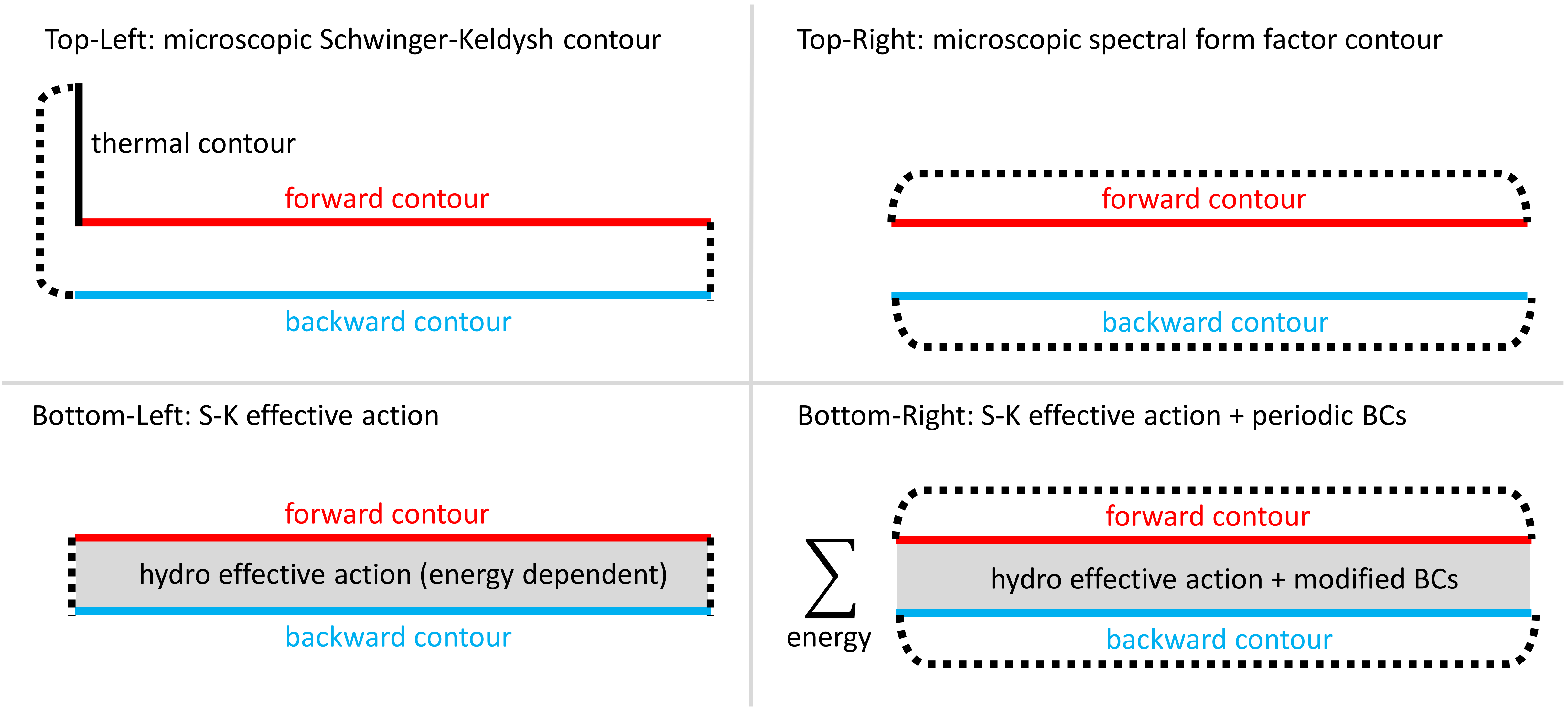
- Uncompleting the square in a-type variable leads to representation of path integral in terms of fluctuating energy diffusion

Ramp from modified CTP on the SFF contours



Assumptions:

- At cutoff scale, same hydro action with modified boundary conditions
- Modified hydro action gives dominant saddle point for SFF
- There is some averaging, e.g. disorder, that connects the contours


Top-Left: microscopic Schwinger-Keldysh contour
thermal contour
forward contour
backward contour
Top-Right: microscopic spectral form factor contour
forward contour
backward contour
Bottom-Left: S-K effective action
forward contour
hydro effective action (energy dependent)
backward contour
Bottom-Right: S-K effective action + periodic BCs
forward contour
$\sum$
energy
hydro effective action + modified BCs
backward contour

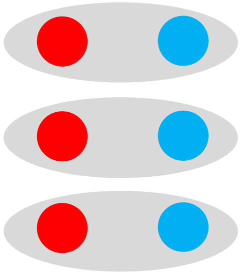
Spatial zero mode

- Times long compared to the longest lifetime: $L = -\phi_a \partial_t \epsilon$
- After regulating the measure, the path integral reduces to an integral over the zero-frequency components

$$\mathrm{SFF} \propto \int d\phi_a(k=0,\omega=0)d\epsilon(k=0,\omega=0) \propto T\int dE$$

- For higher powers, many different ways to connect contours, reproduces expected result from RMT (to leading order)

k=3



e.g. GUE symmety: $\overline{Z^k(Z^*)^k} = k!\mathrm{SFF}^k$

[Winer-S, previously discussed for SYK in Saad-Shenker-Stanford]

Full path integral

$$\mathrm{SFF} = \int \mathcal{D}\epsilon \mathcal{D}\phi_a \exp(iS_{\mathrm{hydro}})$$

$$\mathcal{D}\epsilon\mathcal{D}\phi_a = \prod_x \prod_{\ell=0}^{T/\Delta t - 1} \frac{d\epsilon(x, t=\ell\Delta t) d\phi_a(x, t=\ell\Delta t)}{2\pi}$$

$$S_{\mathrm{hydro}} = \int dV\, dt \left(-\phi_a(\partial_t - D\Delta)\epsilon + i\beta^{-2}\kappa(\nabla\phi_a)^2 \right)$$

eigenvalues of $dt\partial_t$: $T/\Delta t$ complex numbers $i\omega$ obeying $(i\omega + 1)^{T/\Delta t} = 1$

$$\mathrm{SFF} = \prod_k \prod_\omega \frac{1}{i\omega - \lambda_k \Delta t} = \prod_k \frac{1}{1 - e^{\lambda_k T}} \qquad \lambda_k = -Dk^2$$

exactly reproduces prior calculation

[Winer-S]

Elaborations and Examples

- Sound
- Glassy systems
- Late time

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Work with Mike Winer (UMD) [Winer-S 2211.09134]

Sound

$$L_{\mathrm{SK}}^{(\mathrm{sonic})} = \frac{1}{2}\phi_a \left(\partial_t^2 + \frac{2\Gamma}{c^2}\partial_t^3 - c^2 \partial_\mu^2 \right) \phi_r + \frac{2i\Gamma}{\beta c^2}\phi_a \partial_t^2 \phi_a$$

- Sound poles: $\omega \approx \pm ck + i\Gamma k^2$ (+ another rapidly decaying mode)

$$Z_{\mathrm{enhancement}} = \prod_{k>0,\, k^2 \text{ eigenvalue of Laplacian}} \frac{1}{(1 - \exp\{(ick - \Gamma k^2)T\})(1 - \exp\{(-ick - \Gamma k^2)T\})}.$$

- One gets an intricate fractal pattern in 1d
- In d>1, the result depends on the nature of the cavity:
  - If the cavity has an integrable Laplacian, then k is Poisson distributed
  - If the cavity has an RMT-like Laplacian, then the enhancement factor is related to the “single particle” SFF of the cavity!

1d: “Stars over Babylon”

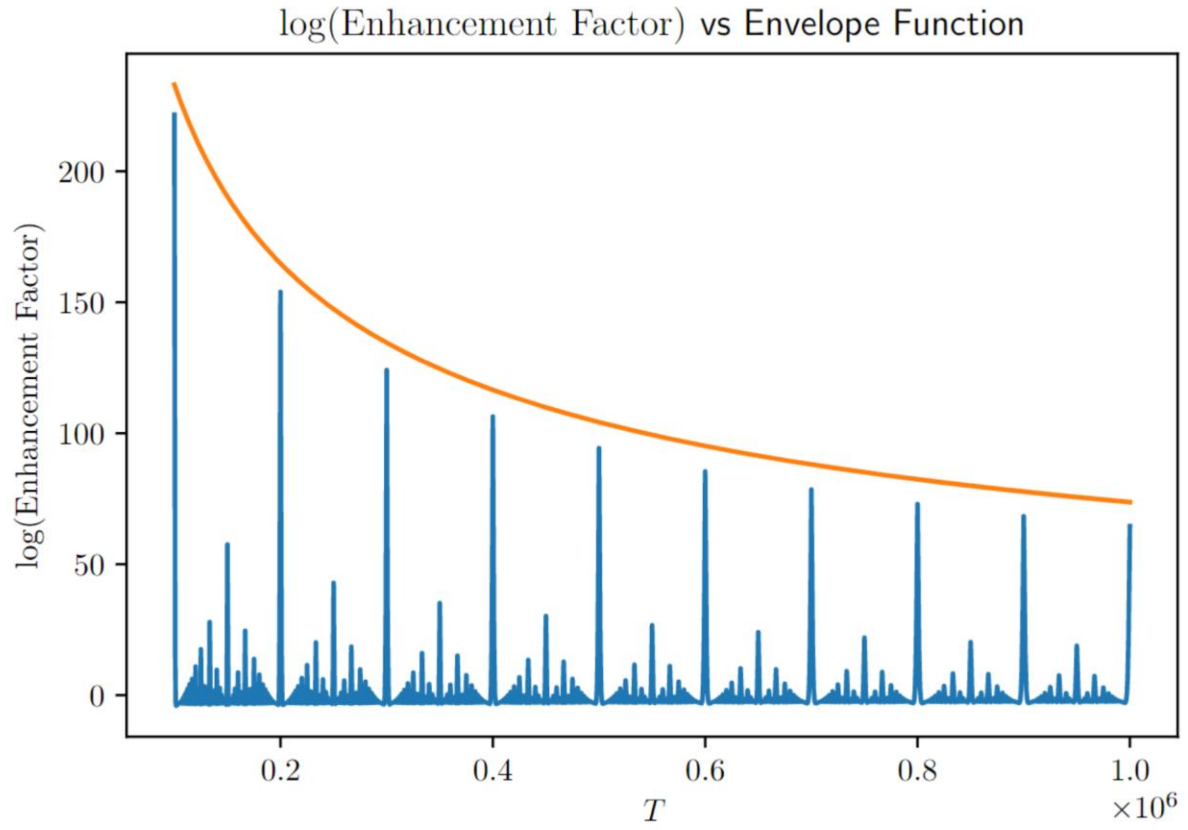


Figure 6: Equation (34) does a good job estimating the envelope for $L = 1,000,000$, $c = 1$, $\Gamma = 1$. The source of the discrepancy comes from approximating the sum in equation (33) as an integral. This approximation is valid in the limit $cL \gg \Gamma$ where many terms contribute to the sum.

Elaborations and Examples

- Sound
- Glassy systems
- Late time

Work with Mike Winer, Richard Barney, Chris Baldwin, Victor Galitski
[Winer-Barney-Baldwin-Galitski-S 2203.12753, Barney-Winer-… 2302.00703]

Glassy dynamics

- In Winer et al. 2203.12753, we computed the SFF for a quantum p-spherical model using path integral methods; we found that the SFF was enhanced by an amount related to the "complexity" of the glass
- In Barney et al. 2302.00703, we introduced a new model, the "block Rosenzweig-Porter model" (BRP), and computed its SFF at all times, generalizing a calculation for RP [Kravtsov et al.]

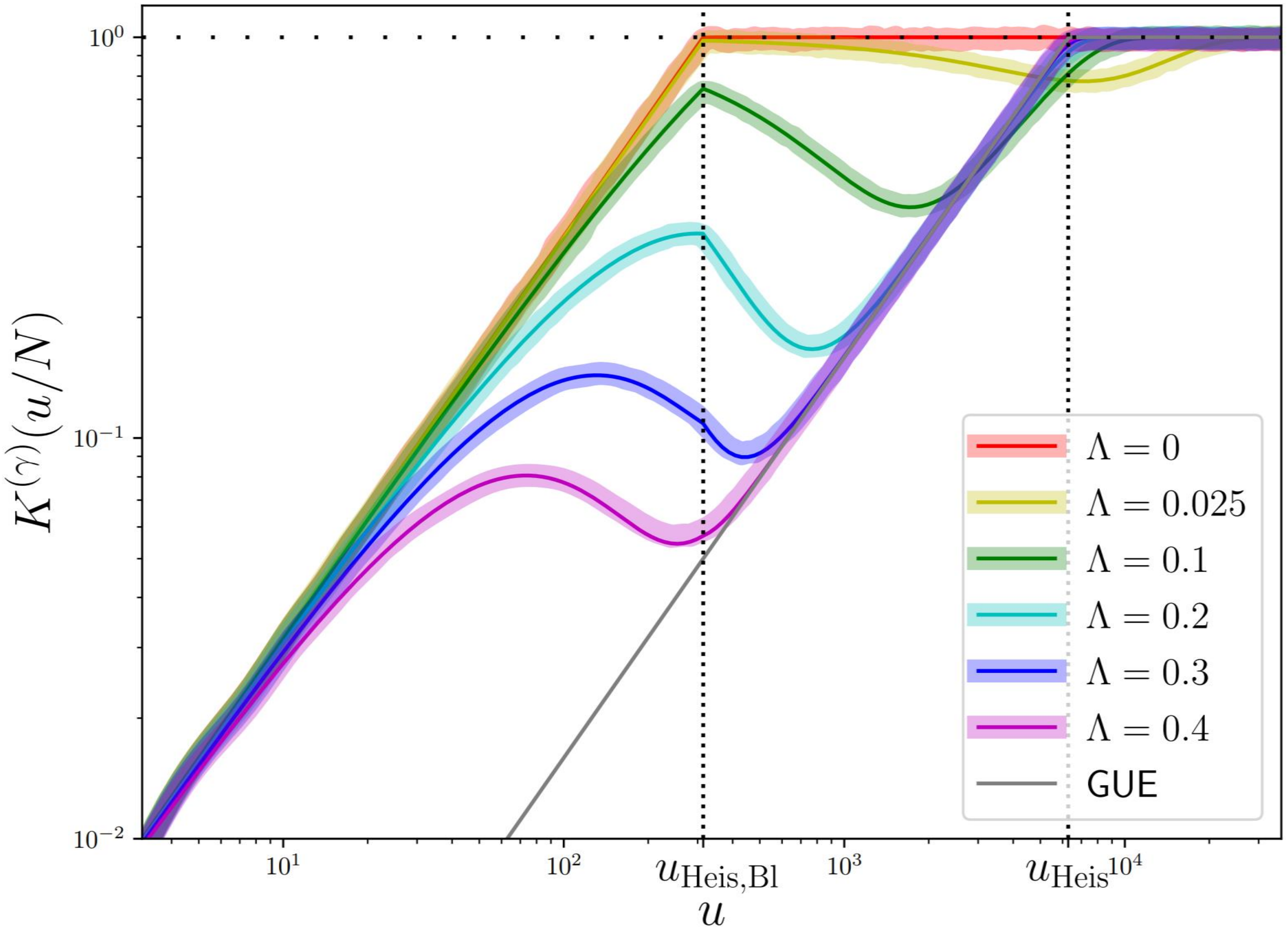
$$H = A + V,$$

$$p_V(V) \sim \exp\left(-\frac{1}{2\sigma}\operatorname{tr} V^2\right), \quad \sigma = \frac{\lambda^2}{N\gamma}.$$

Eigenvalues of A?

- RP: iid
- BRP: eigenvalues of P GUE blocks of size M

N = 1000, P = 20, $\gamma = 1.6$

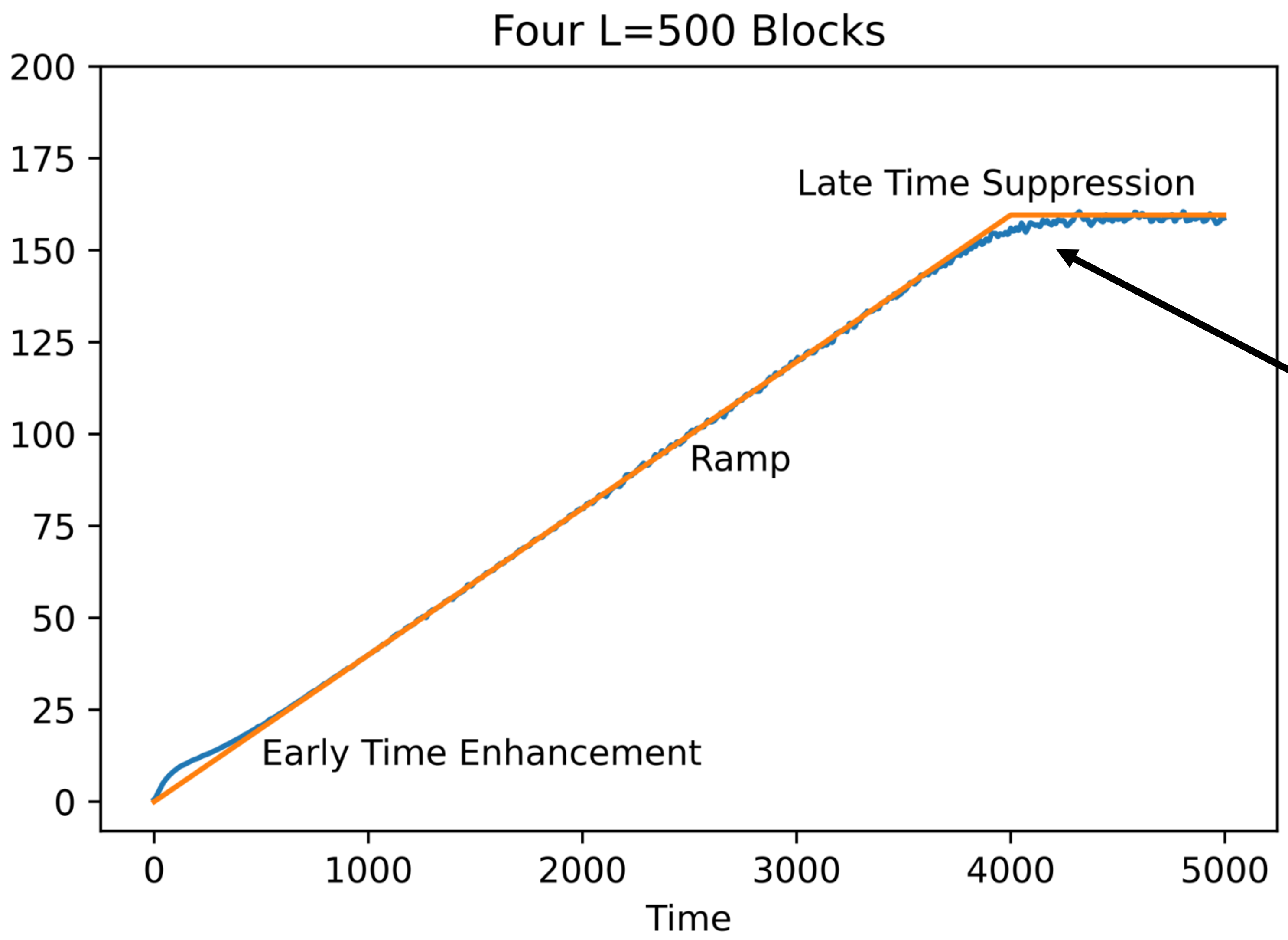


Elaborations and Examples

- Sound
- Glassy systems
- Late time

Work in progress with Mike Winer (UMD)

What about very late time? [preliminary]



We first noticed this in Barney et al. BRP model

SFF Sum Rule

$$\int_{-\infty}^{\infty} [\mathrm{SFF}(T) - L]\, dT = 0$$

- Valid for any system with enough level repulsion → early time enhancements must be "paid for" with a late time suppression
- We conjecture a specific formula for GUE-type problems (derivation in special cases from the Riemann-Siegel lookalike [Berry-Keating])

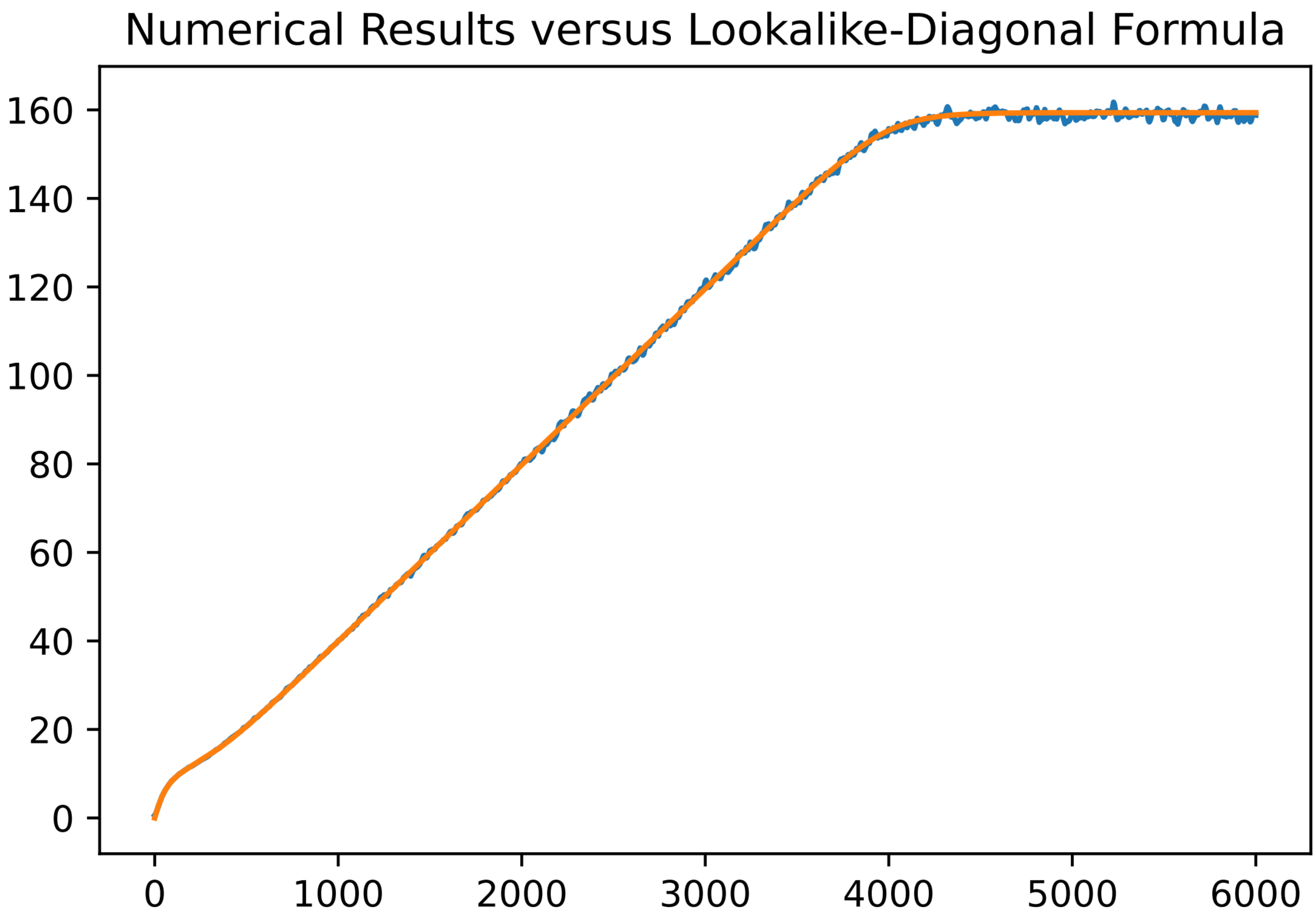
$$\mathrm{SFF}(T) = \mathrm{SFF}_{\text{short time}}(T) + \mathrm{SFF}_{\text{long time}}(T)$$

$$\mathrm{SFF}_{\text{short time}}(T) = \frac{|T|}{2\pi}\left(1 + \sum_n e^{-\lambda_n |T|}\right)$$

$$\mathrm{SFF}_{\text{long time}}(T) = \frac{\lambda_1 e^{-\lambda_1 |T|}}{2} * \frac{\lambda_2 e^{-\lambda_2 |T|}}{2} * \frac{\lambda_3 e^{-\lambda_3 |T|}}{2} \ldots * \mathrm{SFF}^0_{\text{long time}}(T)$$

$$\mathrm{SFF}^0_{\text{long time}}(T) = \begin{cases} 0 & \text{if } |T| \leq 2\pi\hat{\rho} \\ \hat{\rho} - \frac{|T|}{2\pi} & \text{if } |T| > 2\pi\hat{\rho} \end{cases}$$

[Winer-S]


Numerical Results versus Lookalike-Diagonal Formula
160
140
120
100
80
60
40
20
0
0
1000
2000
3000
4000
5000
6000

Outlook

- We are providing tools to compute SFFs in realistic physical systems, including for cases I didn't mention like symmetry breaking (2106.07674) and a Loschmidt version of the SFF (2206.00677)
- The resulting effective theory computes the corrections due to slow modes but also predicts the long-time RMT behavior as well
- There is a surprising link between early time enhancements and a late time suppression near the Heisenberg time, at least for GUE

THANK YOU!