Alterfold Theory and Entanglement Entropy

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Part I: Alterfold Theory

Topological Quantum Field Theory

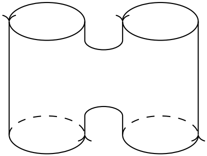
Witten initiated Topological Quantum Field Theory (TQFT) and constructed a 2+1 TQFT using Chern-Simons theory and obtained an invariant of links in 3-manifolds as a path integral [**Wit89**], generalizing the Jones polynomial originated from subfactor theory [**Jon83**, **Jon85**, **Jon87**], and other link invariants from the representation theory of Drinfeld-Jimbo quantum groups [**Jim85**, **Dri86**, **HOMFLY85**, **PT88**, **Kau90**].

Reflection Positivity

Feynman's path integral is a powerful method in physics, but the measure of the path space is only mathematically defined for a few case, such as the remarkable work of Glimme-Jaffe in constructive QFT [**GliJaf87**]. Reflection Positivity of the path integral is a crucial condition to implement the Wick rotation between Euclidean field theory and relativistic field theory, as proved by Osterwalder-Schrader [**OstSch73**].

Atiyah TQFT

Atiyah 88: *Quantum* Physics and *Topology* phenomenon emerging from the continuum limit.
Atiyah's $n+1$ TQFT is defined as a symmetrical monoidal functor from **Cob** to **Vec**, which is a quantum Algebraic Topology approach to TQFT.
Object: n-manifolds without boundary $\rightarrow$ vector spaces
Morphisms: $n+1$ cobordisms $\rightarrow$ linear transformations
The TQFT is called unitary, if the partition function is reflection positive.
In this case, the (finite dimensional) vector spaces are Hilbert spaces.



$$\rightarrow \quad \mathrm{Hom}(V \otimes V, V \otimes V)$$

lower dimensional TQFT

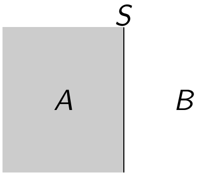
The topological invariant of Witten's 2+1 TQFT can be rigorously defined using the link invariants from quantum groups and the Lickorish-Wallace surgery theory, known as the Witten-Reshetikhin-Turaev TQFT [**ResTur91**].

The Turaev-Viro-Barrett-Westbury 2+1 TQFT from a spherical fusion category [**TurVir92**, **BarWes96**] is a state sum construction over a triangulation, which is a combinatorial analogue of path integral.

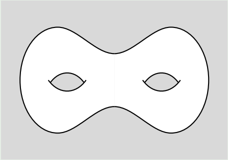
Alterfold TQFT

An n-alterfold consist of

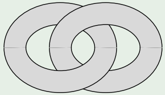
- A closed oriented n-manifold M;
- A separating hyper surface $S \subset M$;
- S separate M into A-B colored regions. (S may not be connected.)



Example



B-colored handlebody in A



A-colored Hopf Link in B

An alterfold TQFT is a functor from the cobordism category of alterfolds to the category of vector spaces. It reduces to Atiyah's TQFT when the manifold is fully B-color.

2+1 Alterfold TQFT

In joint work with S. Ming, Y. Wang and J. Wu [**LMWW23a**, **LMWW23b**], we constructed a 2+1 alterfold TQFT from a spherical multi-fusion category $\mathscr{C}$. Moreover, we proved that the 2+1 alterfold TQFT contains both

- Turaev-Viro-Barrett-Westbury TQFT of $\mathscr{C}$
  (blow up the skeleton of the triangulation to B-color handles)
- Witten-Reshetikhin-Turaev TQFT of its Drinfeld center of $\mathscr{C}$
  (blow up the framed links to A-color handles)

Moreover, the two sub TQFT are identical on Atiyah's TQFT of cobordisms.

$$\begin{array}{ccc} \text{WRT TQFT} & \subset & \text{Alterfold TQFT} \\ \cup & & \cup \\ \text{Atiyah TQFT} & \subset & \text{TVBW TQFT} \end{array}$$

1. Walker 91,03 Turaev 94, Robert 95 for MTC;
2. Kawahigashi-Sato-Wakui 05 for unitary SFC;
3. Turaev-Virelizier 17, Balsam-Kirillov 10 for SFC.

Meanings of A/B colors

Take $Z(M_A) \equiv 1$, namely the partition function of any A-color manifold M is constant 1. Then the partition function is irrelevant to the A-color part, which encodes to the surgery theory in the 2+1 TQFT.
The B-color encodes the 1-dim higher center.
We call A the trivial color and B the bulk color.
If we run the alterfold construction twice, then the $n+2$ manifold invariants is trivial, due to the triviality of A-color. This phenomenon has been considered as the center of the center is trivial.

Alterfold Construction

In **[Liu 24]**, we provide a functional integral construction of topological quantum field theory in any dimension.

n-dim Lattice Hamiltonian H
→ higher state Z
→ higher (operator) algebras
→ higher representations
→ higher categories
→ n+1 TQFT with space-time boundary (for B-color manifolds)
→ unitary n+1 alterfold TQFT

The TQFT is topological, because the local Hamiltonian is independent of the matrix of the manifold.

The categorical symmetry is an emergent symmetry. (The classical group symmetry is invertible. The categorical symmetry may not be invertible.)

TQFT describes the emergent global symmetry of the lattice model.

Function Integral Construction of TQFT

Theorem (Liu 24, arXiv:2409.17103v1)

Suppose Z is a linear functional on a random lattice model with support manifold S^n over the field $\mathbb{C}$, satisfying the three conditions

1. *(RP) reflection positivity;*
2. *(HI) homeomorphic invariance;*
3. *(CF) complete finiteness. (finite entanglement rank)*

Then we obtain an $n+1$ unitary alterfold TQFT.

Remark: The lattice model is mathematically formulated by labelled regular stratified piecewise-linear manifolds. The regular condition is crucial to ensure the transversal theorem in the alterfold construction.

An answer to Wen's question

The three conditions (RP) (HI) (CF) of the linear functional Z provides a characterization of topological orders from its ground states which is an answer to the fundamental question of XG Wen.
Our functional integral construction suggests a mathematical definition of unitary/spherical n-categories.
In our approach, the unitary n-category is an emergent quantum symmetry, instead of an axiomatized input data.
If we take the linear functional Z to be the evaluation map of the spherical multi-fusion category $\mathscr{C}$, then we recover the construction of 2+1 alterfold TQFT in [**LMWW23a**, **LMWW23b**].

Topological Orders

In joint work with A. Jaffe **[L-Jaffe 20]**, we proved that the Hamiltonian from the Levin-Wen model is RP. (The HI property is obvious.)
Then the linear functional

$$Z = e^{-\beta H} |\vec{0}\rangle ,$$

is RP for any inverse temperature β.
When $\beta = \infty$, Z is the ground state of H, which is a non-trivial topological order.
In our theoretical approach, we will also obtain a unitary 2-category from Z at a finite temperature, and furthermore a 2+1 alterfold TQFT assuming CF.

Motivating Questions and Conjectures

Question: What are the categorical symmetry and TQFT at finite temperatures?

Question: What is different between the theory at $\beta = 0$ and $\beta > 0$?

Conjecture (Entanglement Rank Conjecture)

The entanglement rank of $Z = e^{-\beta H} |\vec{0}\rangle$ between two half n-discs for any boundary condition on S^{n-1} is independent of β, $0 < \beta < \infty$. At $\beta = \infty$, the rank is smaller, due to additional emergent symmetry.

Conjecture (Robustness of Emergent Symmetry)

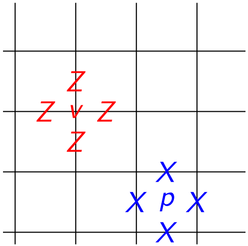
The n-categorical symmetry (and n+1 TQFT) from $Z = e^{-\beta H} |\vec{0}\rangle$ are isomorphic for $0 < \beta < \infty$. At $\beta = \infty$, we obtain a sub n-category.

Remark: The second conjecture is stronger than the first, because the entanglement rank equals to the dimension of the categorical hom space for a given boundary condition.

Part II: Entanglement Entropy
Joint with Zishuo Zhao

Toric Code

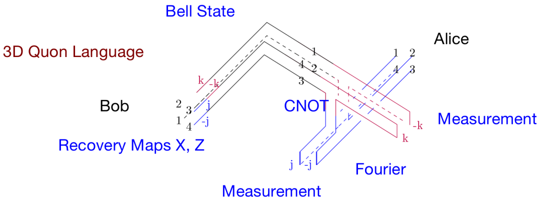
Kitaev proposed the toric code to study topological orders. **[Kit97]**

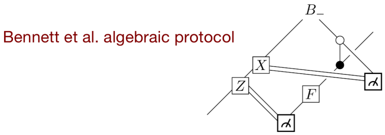


Each edge in the $D \times D$ lattice has a physical qubit.
Stabilizers: $A_v = \prod_{v \in \partial e} Z_e$, $B_p = \prod_{e \in p} X_e$.
Hamiltonian $H = -\sum_v A_v - \sum_v B_p$
The entanglement entropy of its ground state includes two terms: one follows the area law of the boundary; the other is topological.

Quon Language for Quantum Information

In joint work with Jaffe and Wozniakowski **[Liu-Jaffe-Wozniakowski 17]**, we introduced the Quon language to study quantum information, which can also be used to study the toric code.
In quon language, tensor networks blow up to handlebodies with decorations on the boundary surface.

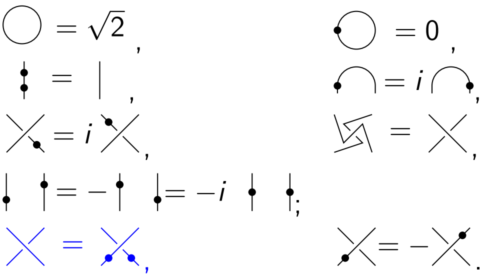




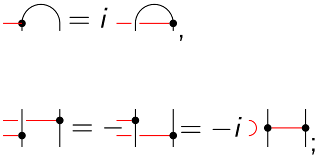
Skein Relations

The braid satisfies Reidemeister moves of type I, II, III.

The charge behaves like a Majorana fermion:

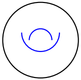


Topological Relations



(Last two only for qubits)

String-Genus Relation:

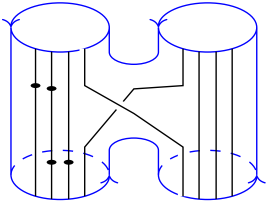


$$= \frac{1}{\sqrt{2}}.$$

Quon Language in Alterfold Theory

If we start with the linear functional Z on S^2 with embedding strings, such that each closed string contributes to the value $\delta = \sqrt{2}$. Then Z verifies the three conditions three properties (RP), (HI), (CF).

We obtain a 2+1 alterfold TQFT, generalizing the Quon language. The handlebodies in the quon language are B-color 3-manifolds in alterfold TQFT. The skein relations of string decorations can be derived from the kernel of the linear functional Z.

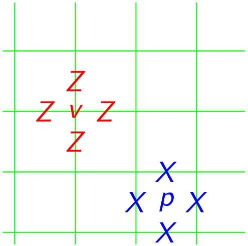


The alterfold TQFT involves 3-manifolds beyond handlebodies, which provide brand new insights and computational tools to study the toric code and quantum information!

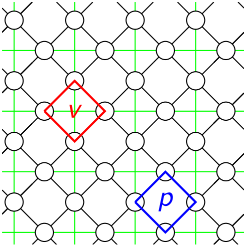
Alterfold Presentation of Ground States

The 2^2 dimensional ground state space of the toric code on the torus T can be expressed by a 3-alterfold:
B-color manifold$=T \times [0,1]$,
Input time boundary $T \times 0$,
Output time boundary $T \times 1$ with lattice decorations.
When the input $T \times 0$ is filled by the surface T without decoration, the ground state is the canonical one, given by "the sum of contractible loops".

Periodic Square Lattice

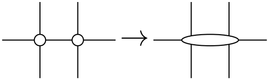


Lattice Decorations on $T \times 1$

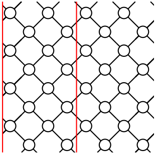


Equivalent Presentations for Entanglement

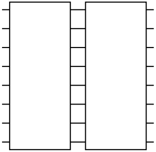
Local unitary transformation:



Ground State



Equivalent Ground State



From this equivalent presentation, one can easily compute the entanglement entropy and transparently see the area law and topological entanglement entropy.

The method works for any entanglement parition over any surface. The topological entanglement entropy is determined by the topological invariant of certain three manifolds.

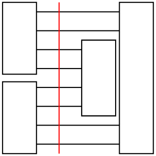
Connectedness Entanglement Entropy

In the entanglement entropy of the ground state, we find the third term Connectedness Entanglement Entropy (CEE) related to the connectedness of the lattice, in addition to the area law and the topological entanglement entropy.

Theorem (Liu-Zhao 25)

$$CEE = -r \log 2.$$

Example: $CEE = -2 \log 2$



$$r = dimKer \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} - dimKer \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = 2$$

Entanglement Entropy at Finite Temperatures

At a finite inverse temperature $0 < \beta < \infty$, we can present the quantum thermal state $Z = e^{\beta H} |\vec{0}\rangle$ as a PEPS in the alterfold TQFT.

Furthermore, for an entanglement separation C and D of the surface with the common boundary ∂, we can block diagonalize the reduced density matrix of Z as a direct sum of 2 by 2 matrices, indexed by neutral bi-partitions S and S^c of the boundary points.

We can compute the eigenvalues of the reduced density matrix explicitly, when the volume of D-region $|D| \to \infty$ and C is connected,

$$\left(\frac{\theta}{1+\theta^2}\right)^{|\partial\mathcal{P}|} \frac{1}{2}\left(\eta(|S|) \pm \sqrt{\eta(|S|)^2 + 4\left(\frac{2\theta}{1+\theta^2}\right)^{2|C|} - 4}\right).$$

where $\theta = tanh\frac{\beta}{2}$, and $\eta(|S|) = \theta^{|S|-|S^c} + \theta^{|S^c|-|S|}$. The Rényi entropy is

$$S_2(\rho_A) = |\partial| \log 2 - \log 2 + \frac{|A|}{2}(1-\theta)^2 + O((1-\theta)^3).$$

Robustness of CEE

At a non-zero temperature, the topological entanglement entropy will suddenly disappear when $|C|, |D| \to \infty$, **[Castelnovo-Chamon 07]**.

Theorem (Liu-Zhao 25)

The connected entanglement entropy (CEE) of the thermal pure state $Z = e^{-\beta H} |\vec{0}\rangle$ *is robust near zero temperature:*

$$S_2(\rho_A) = |\partial| \log 2 + CEE + \frac{|C|}{2}(1-\theta)^2 + O((1-\theta)^3).$$

The first term follows the area law; the second is CEE, the third follows the volume law.

The computation of the entanglement entropy rely on new insights and computational tools in the alterfold TQFT at a finite temperature.

Outlook

For the toric code, we show that the unitary 2-category arisen from $Z = e^{-\beta H} |\vec{0}\rangle$ is the Ising category for $\beta = \infty$; and it is the graph planar algebra **[Jones99]** of the Ising category at $0 < \beta < \infty$.
This example verifies the two conjectures: (1) Entanglement Rank Conjecture and (2) Robustness of Emergent Symmetry.

Conjecture

For the Levin model of a unitary fusion category $\mathscr{C}$, we conjecture that the emergent unitary 2-category at $0 < \beta < \infty$ is the graph planar algebra of $\mathscr{C}$.

It is known that $\mathscr{C}$ is the flat part of a flat connection on the graph planar algebra in subfactor theory.
From this point of view, the theory at $\beta = \infty$ comes from the fixed points of an additional categorical gauge symmetry. This additional categorical symmetry essentially produces non-trivial topological orders and topological invariants of $n + 1$ manifolds.

Functional Integral Construction of TQFT
ArXiv:2409.17103v1
Entanglement Entropy at Finite Temperatures
to appear
Thank you for your attention!