A Potential Method for Logical Gates via Anyon Permutation

C. N. Yang Institute for Theoretical Physics,
Stony Brook University

Zijian Song

Joint work with Yabo Li (NYU -> Purdue)
arXiv: 2602.10110,
and other ongoing works

Mathematical Picture Language Seminar

Scope of this talk

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- **Topological codes** that encode logical information in their **ground-state subspace**.

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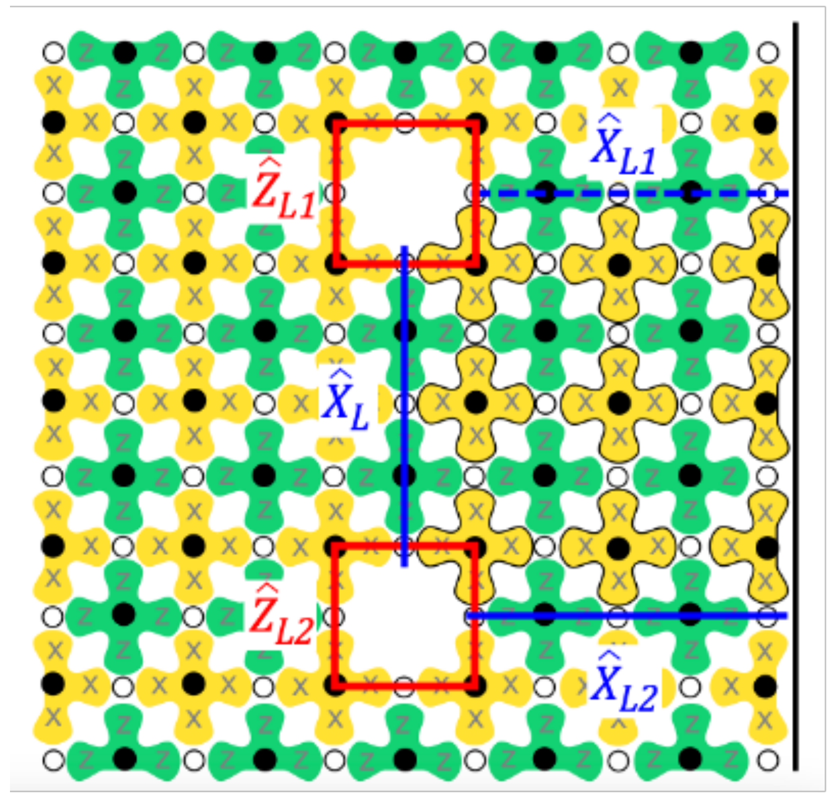
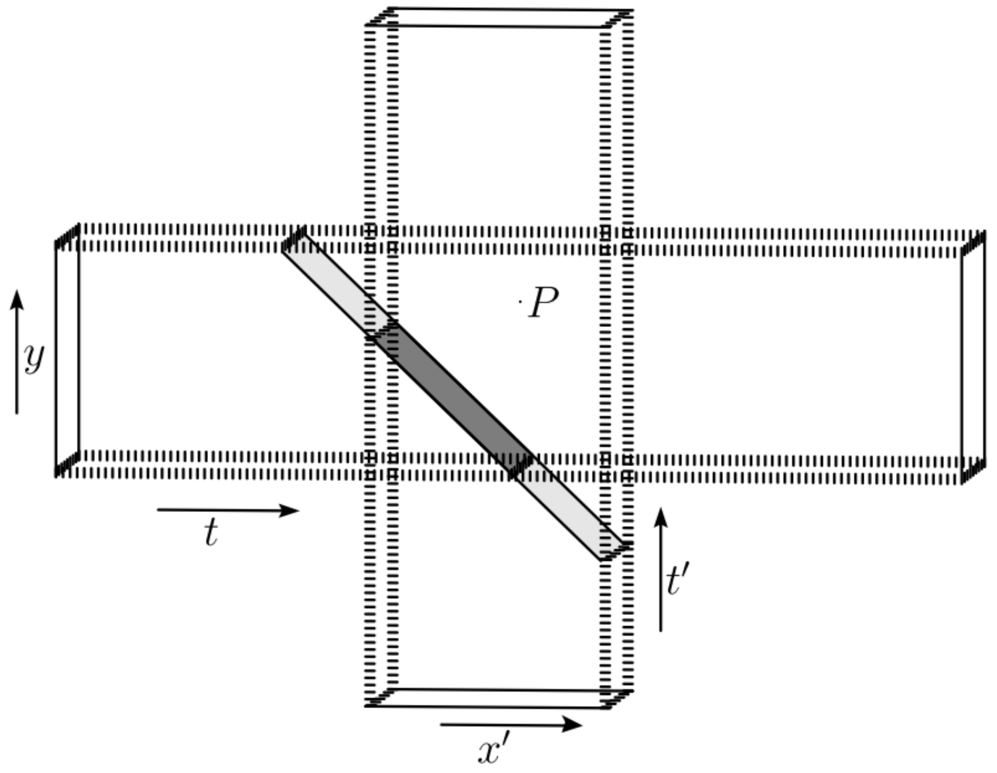
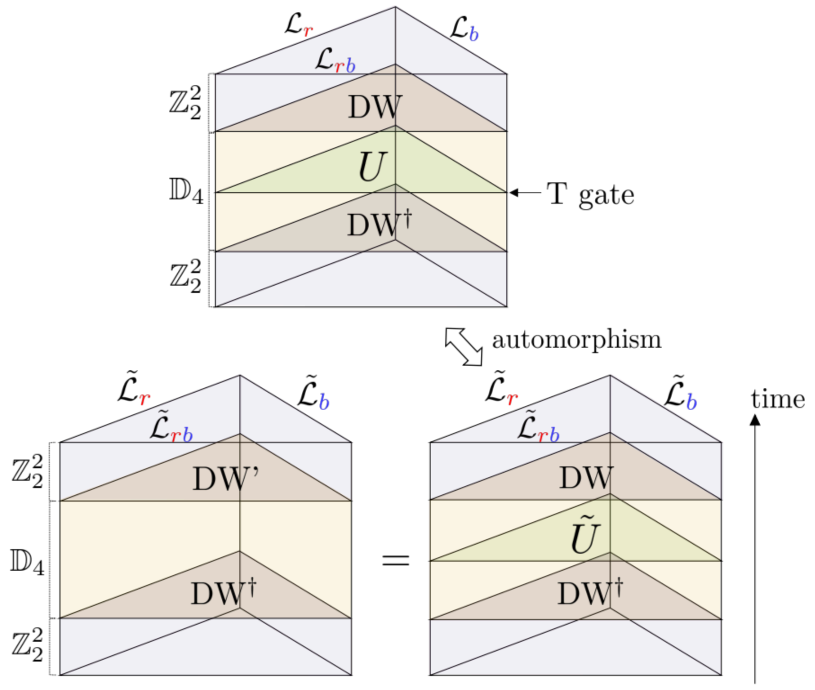


Fowler et al. 2012



Brown 2020



Kobayashi et al. 2025

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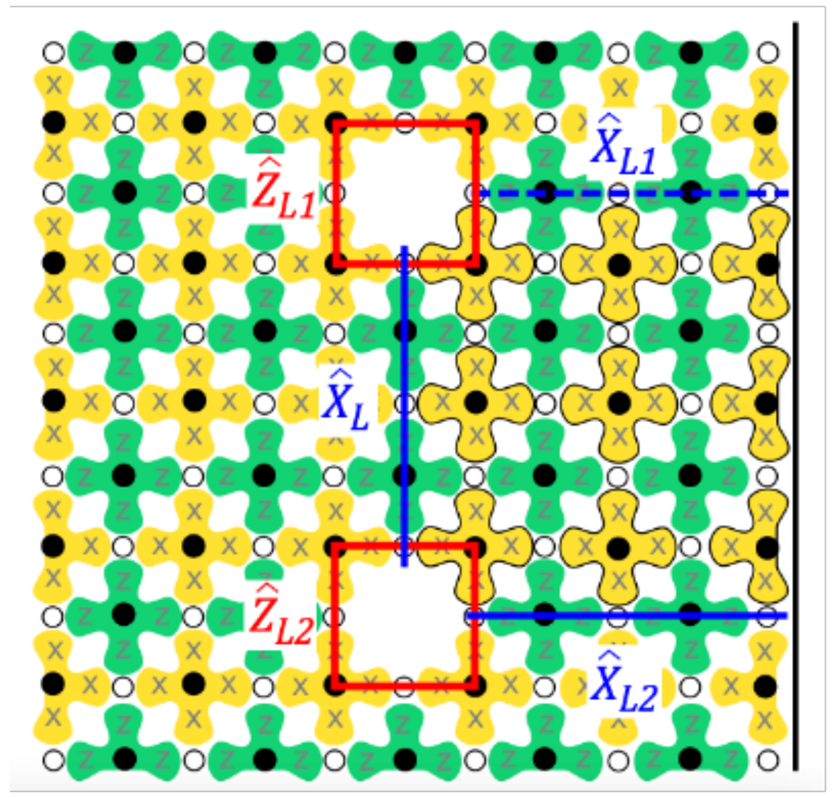

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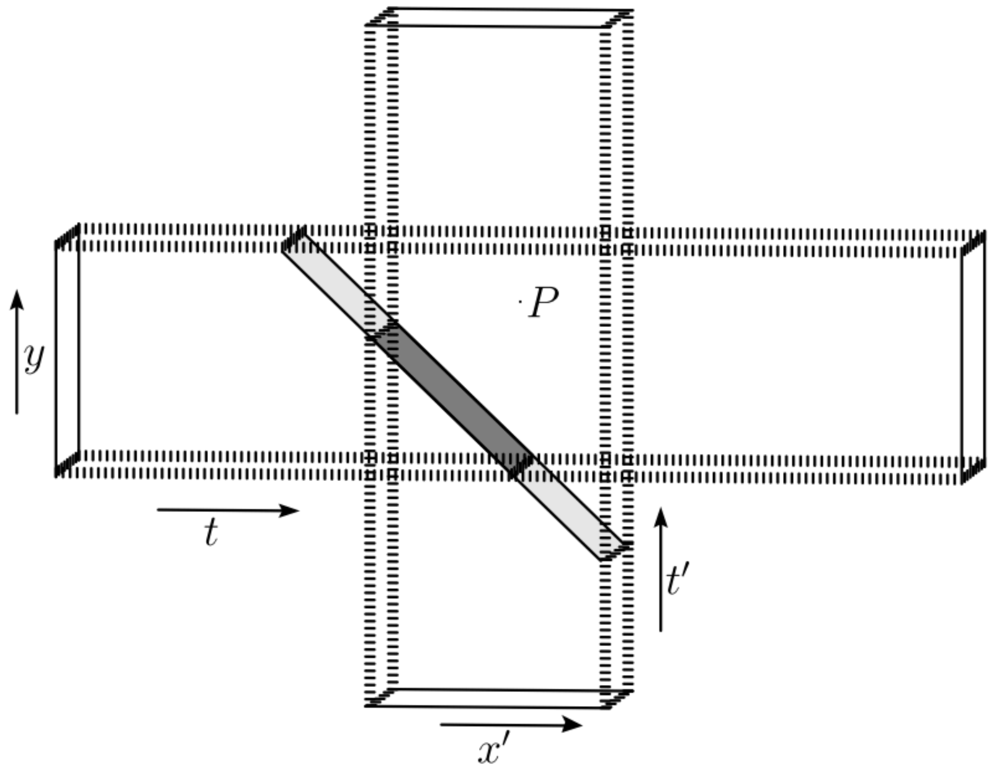

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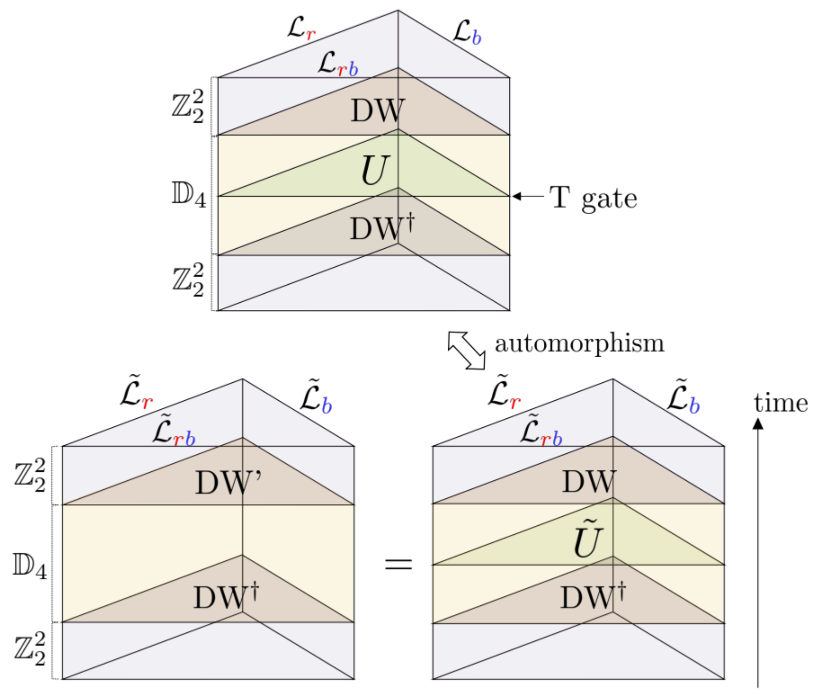

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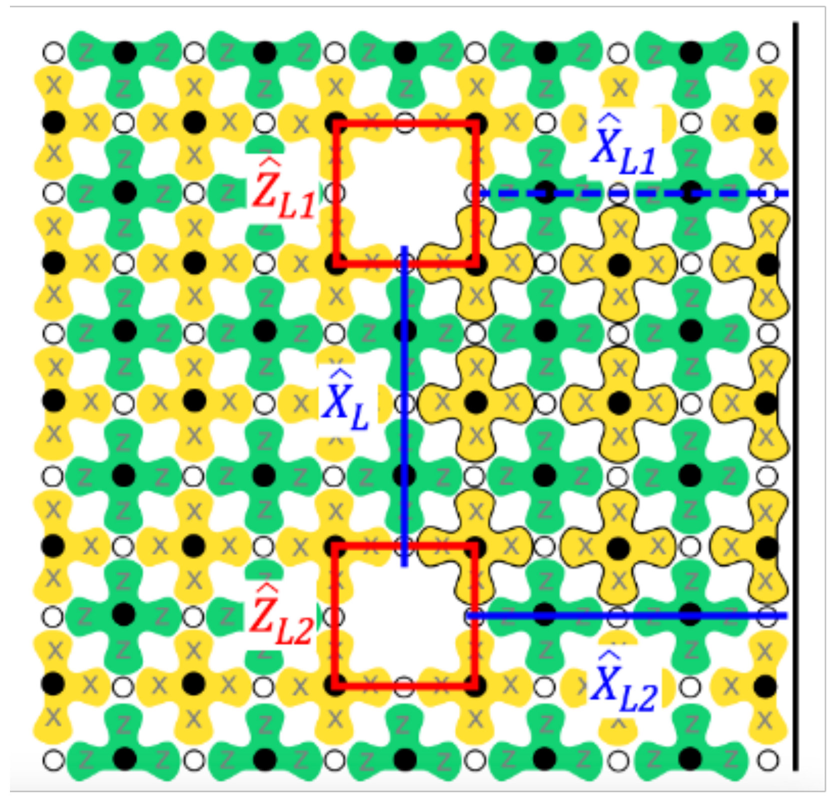

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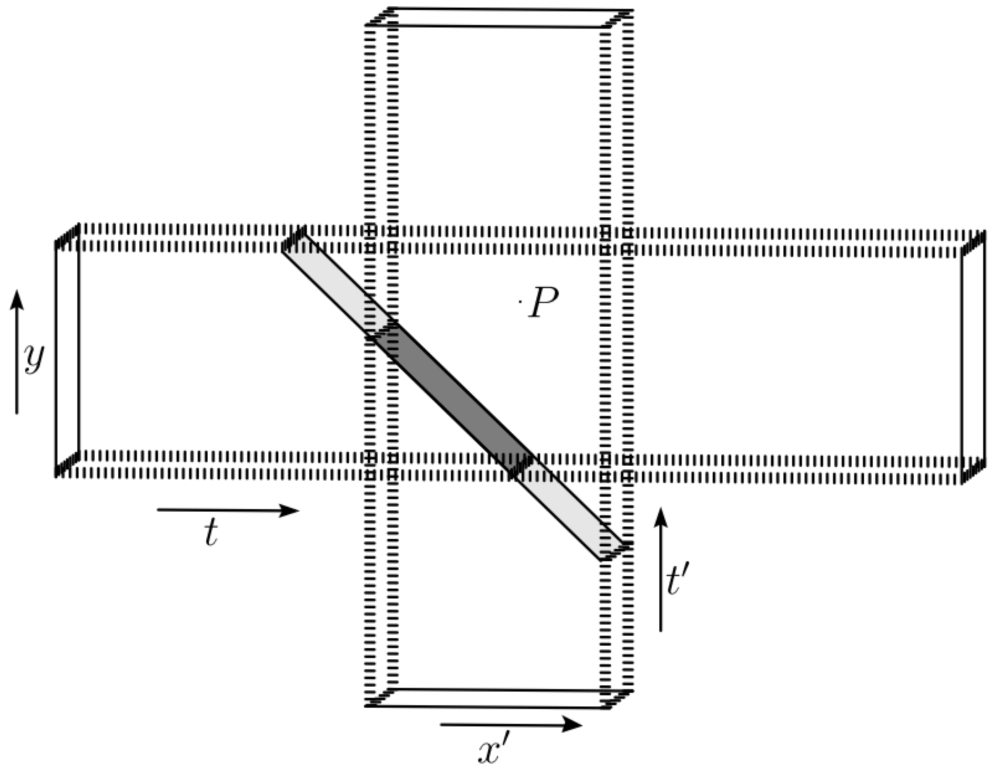

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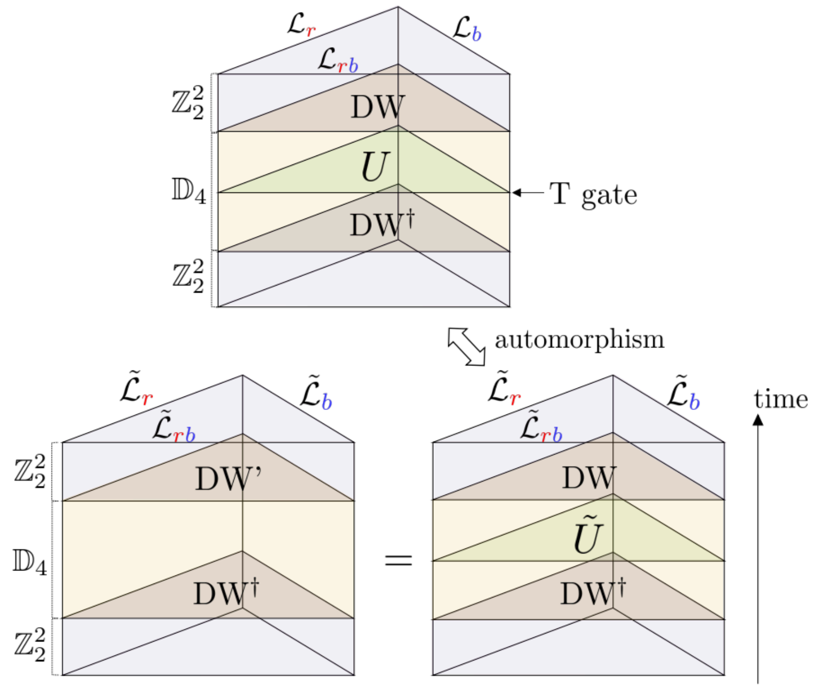

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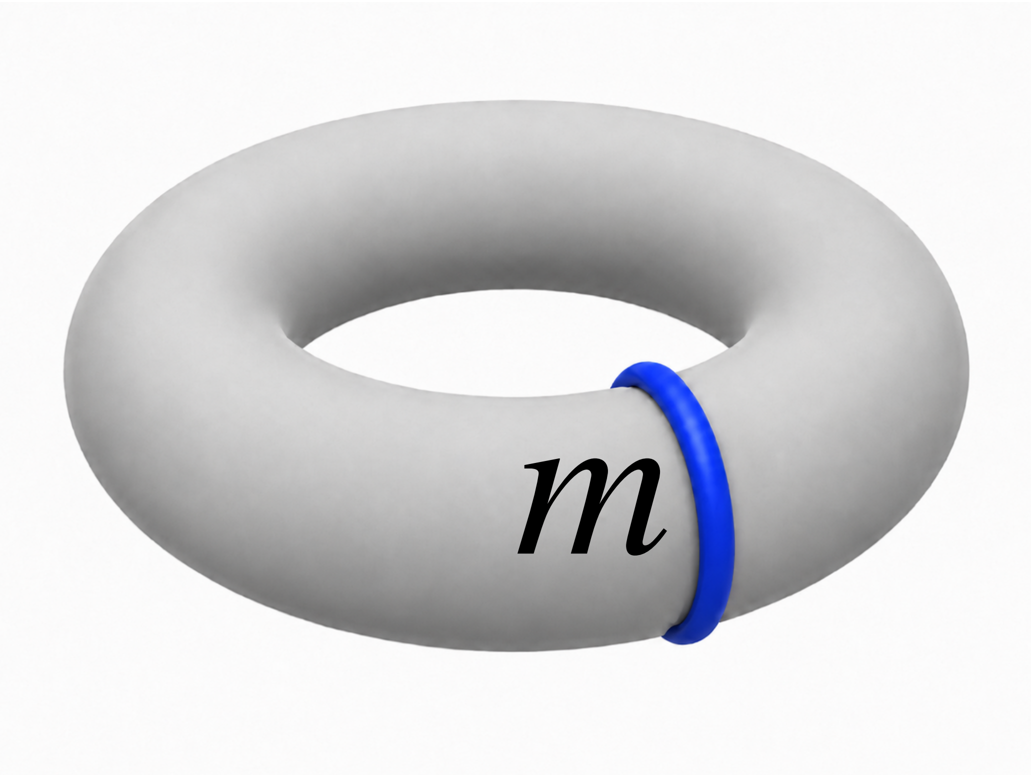

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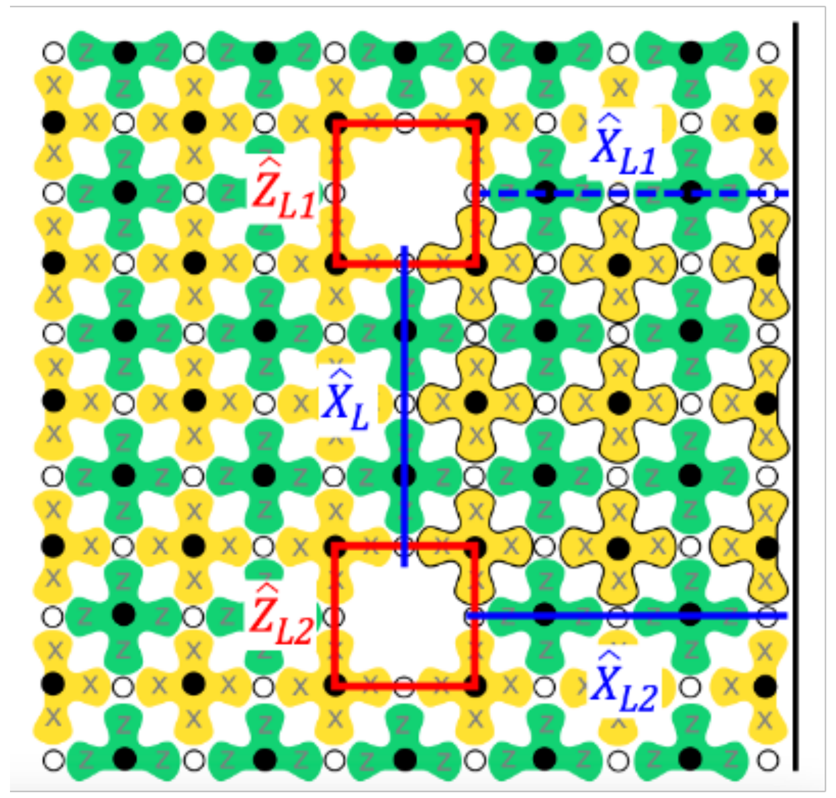

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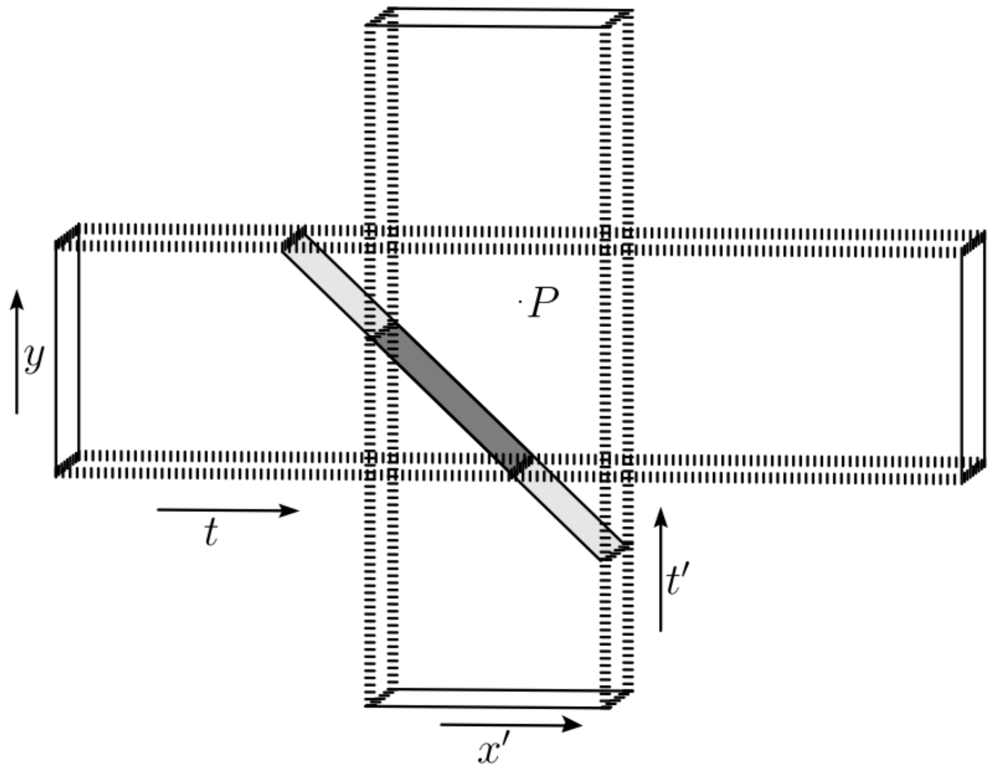

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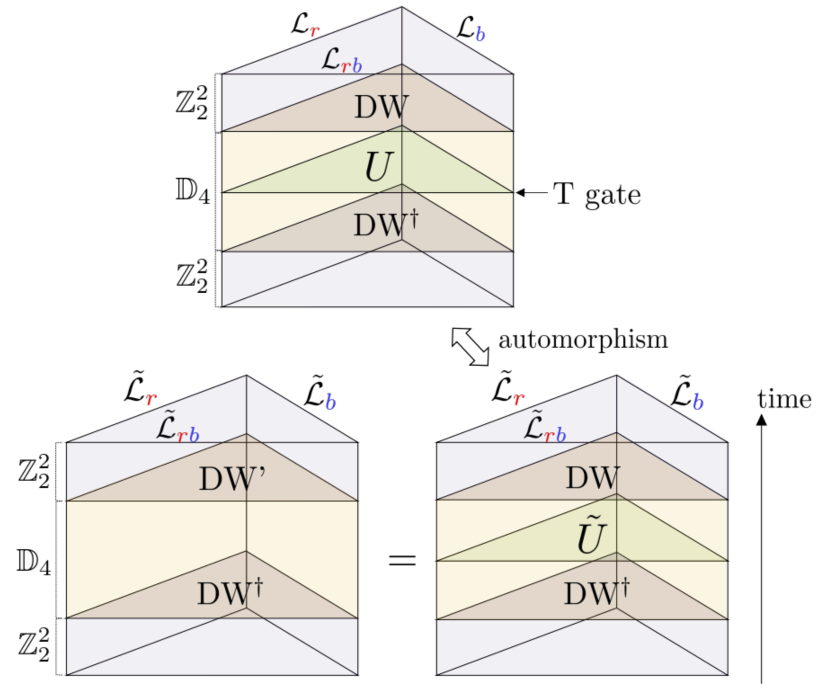

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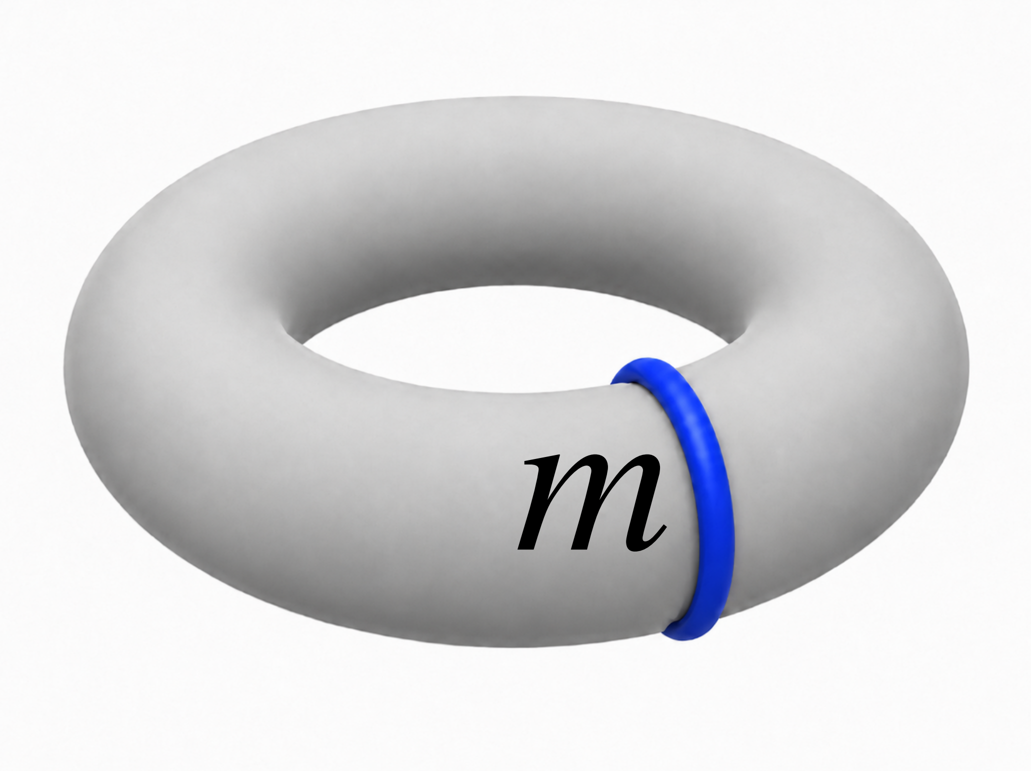

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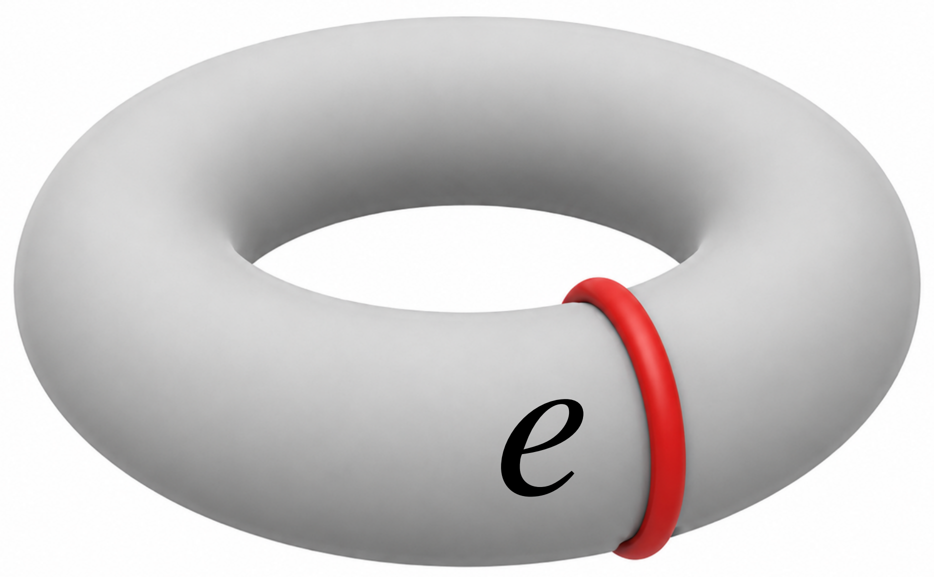

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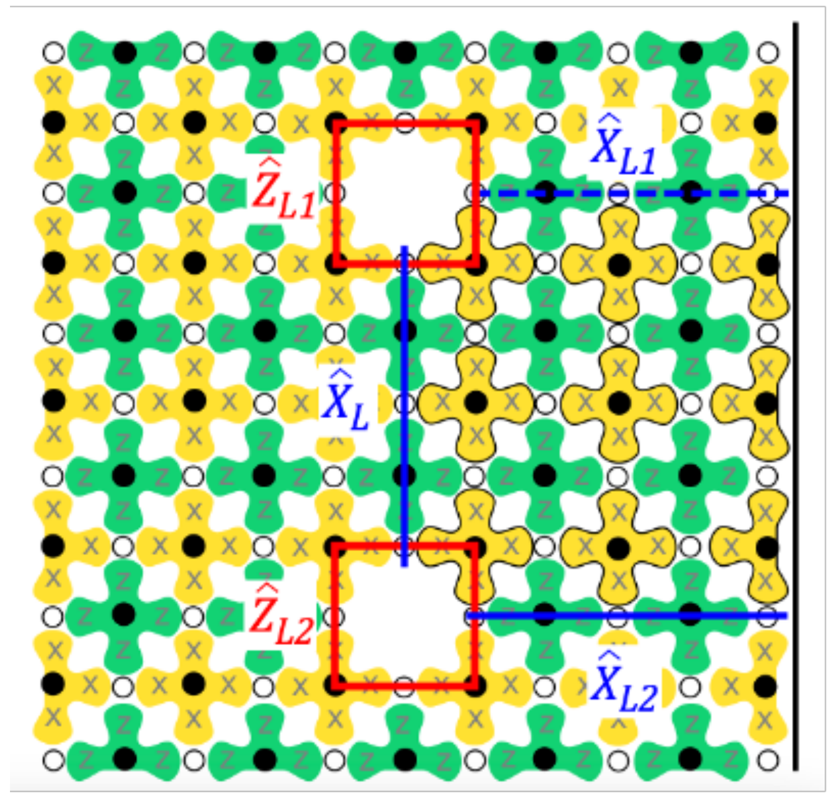

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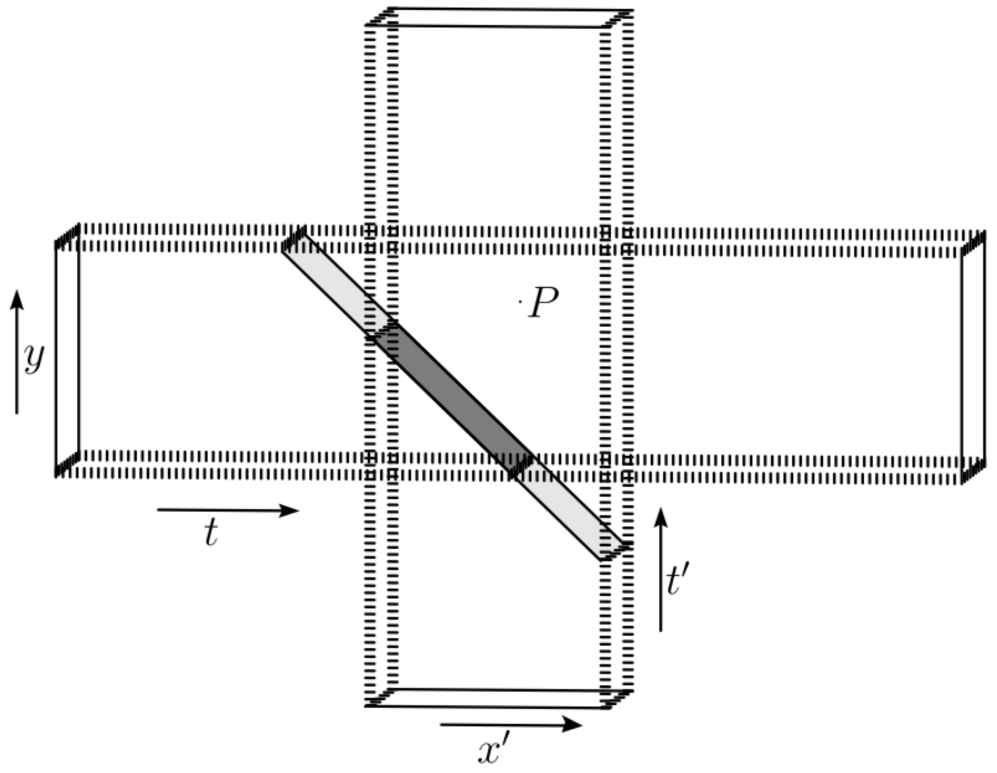

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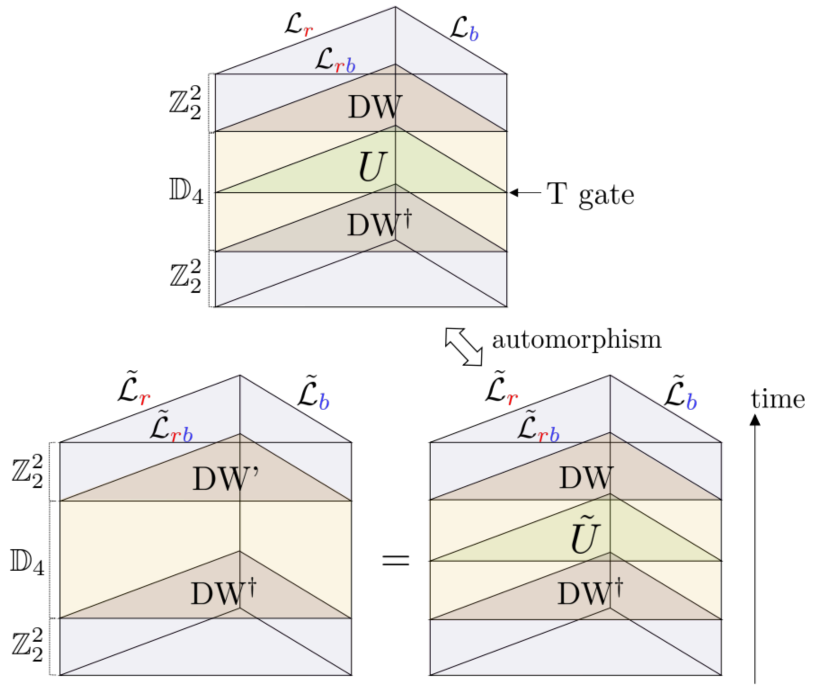

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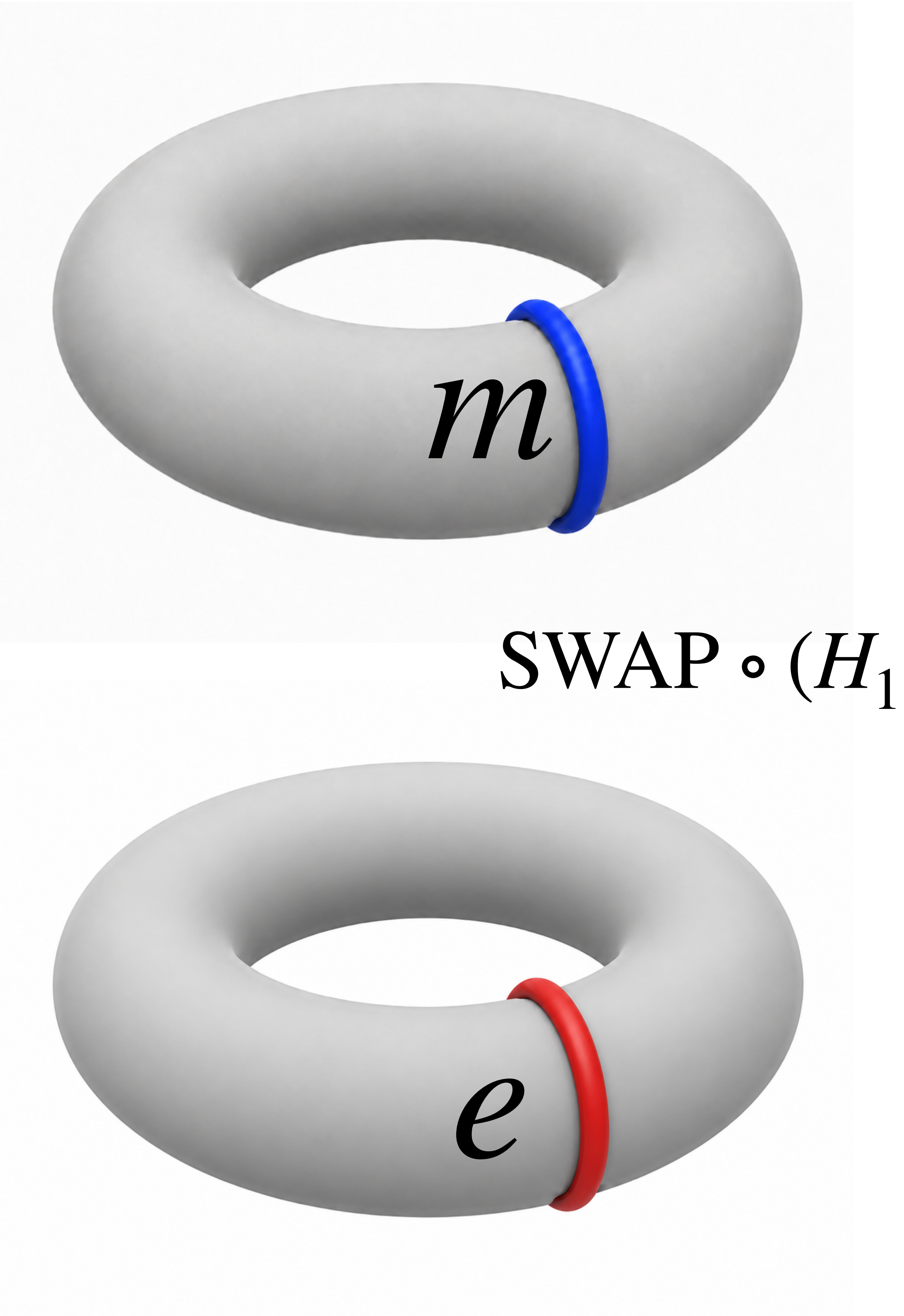

$$|1\rangle \otimes |0\rangle$$

$$\mathrm{SWAP} \circ (H_1 \otimes H_2)$$

$$|+\rangle \otimes |-\rangle$$

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Take-home messages

All anyon permutations* in $D(G)$ quantum double models can be realized by ***constant-depth local unitary*** circuits.

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1D Self-Duality for G Symmetry $\xleftrightarrow[\text{Holography}]{\text{Topological}}$ 2D Anyon Permutation

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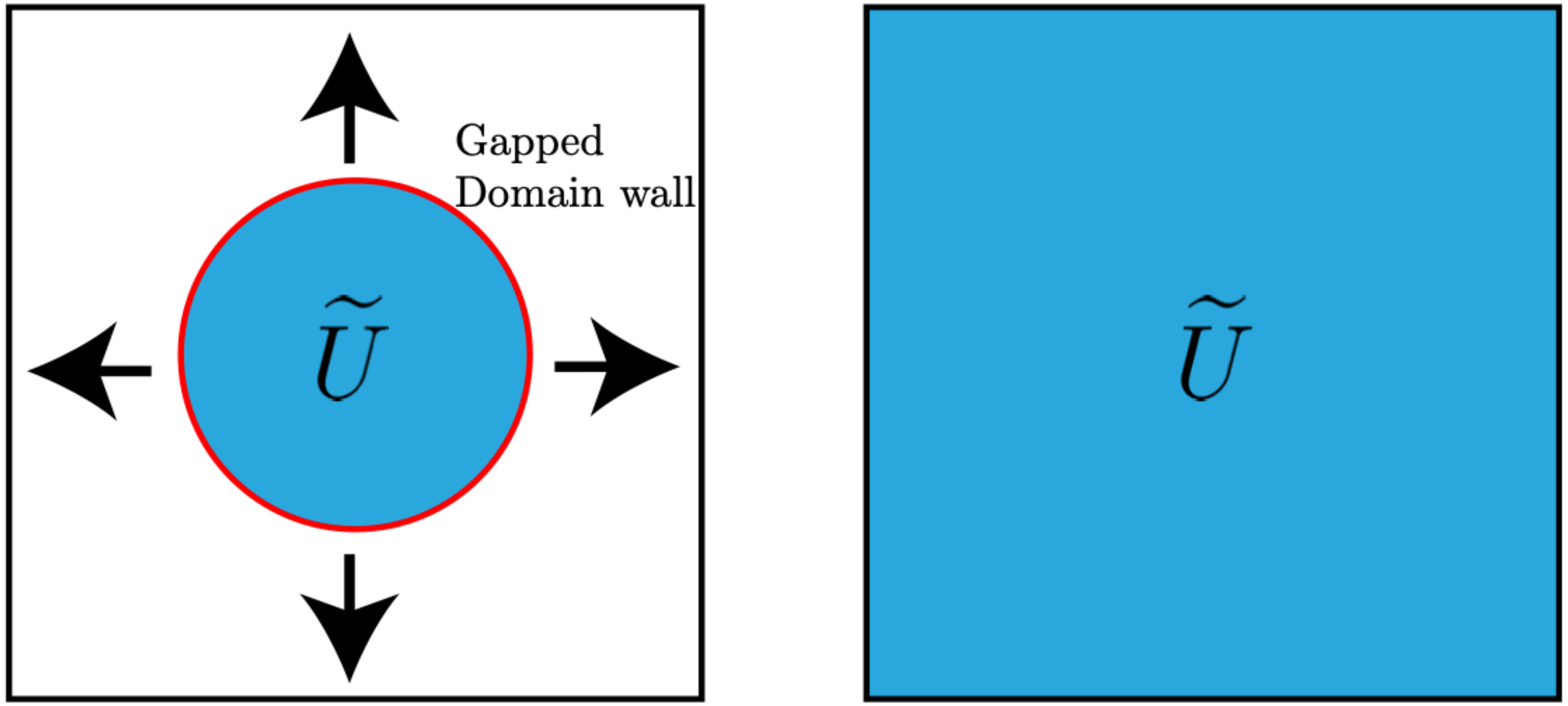
Li, ZS (2026)

Intuition

Realizing anyon permutations is equivalent to sweeping the associated *invertible domain walls*



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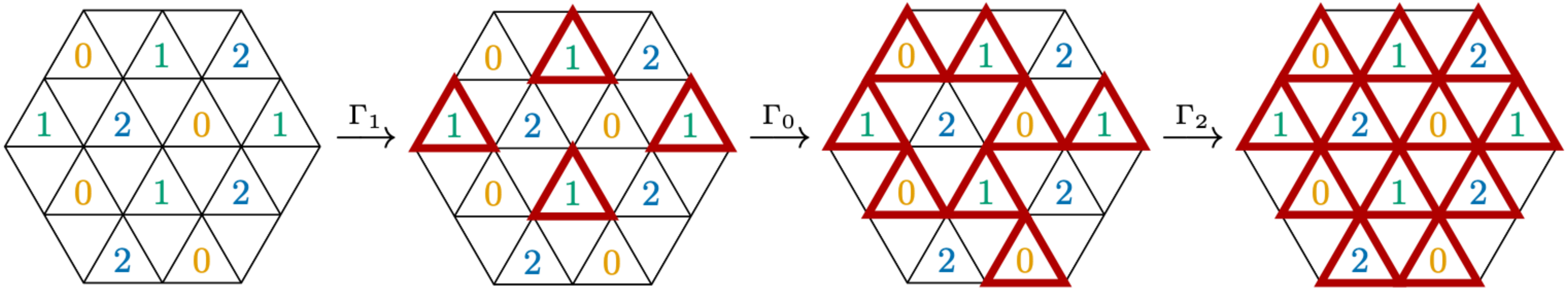
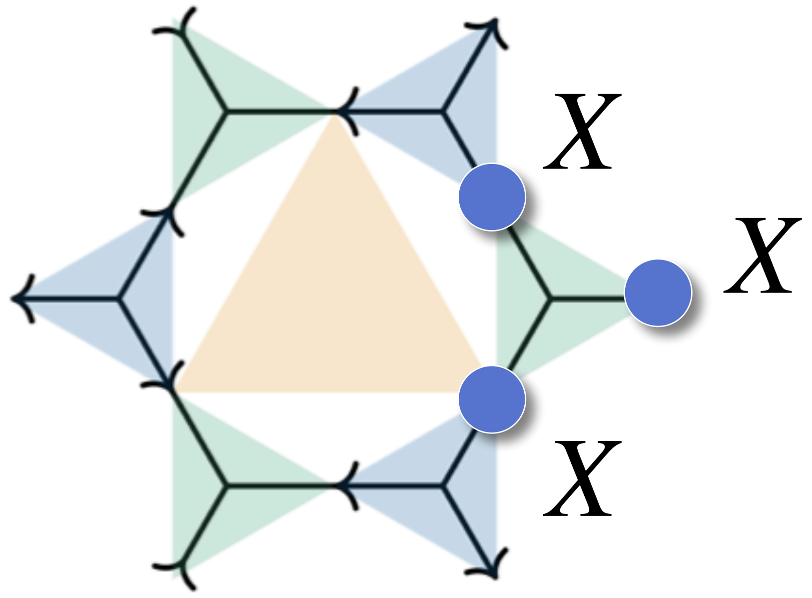
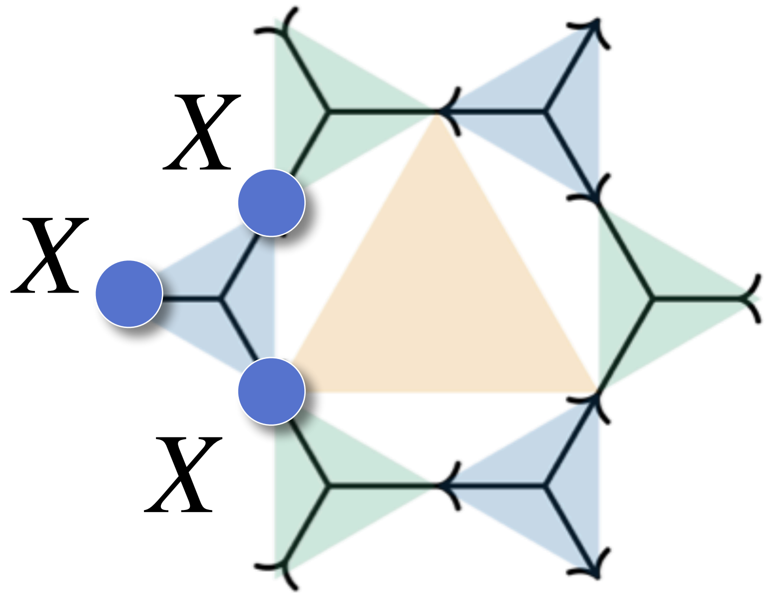


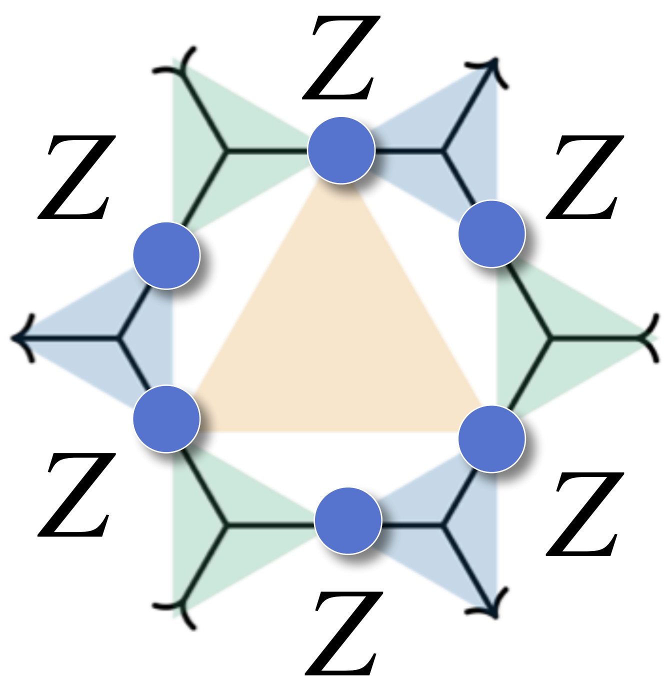
Aasen, Wang, Hastings (2022)
Li, ZS (2026)

Realizing e - m duality in the toric code model

- Toric code stabilizers







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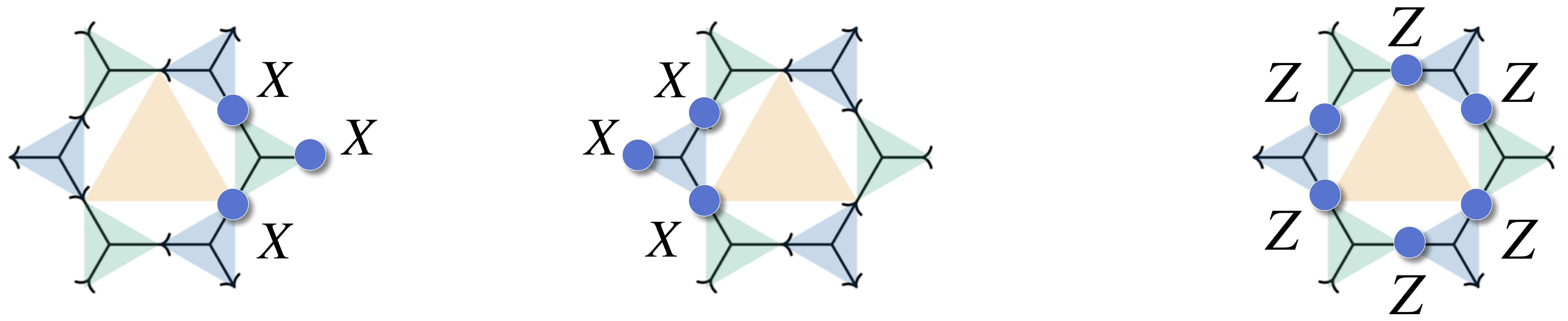
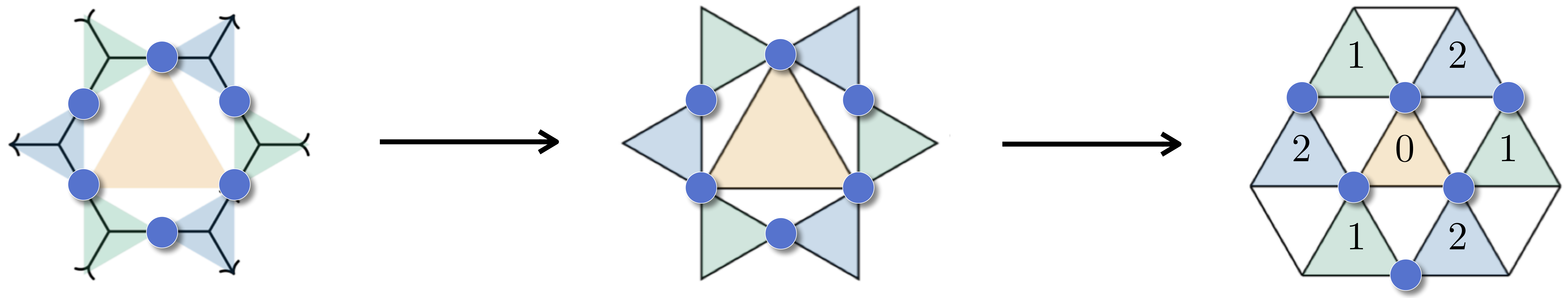


- Applying lattice deformation:



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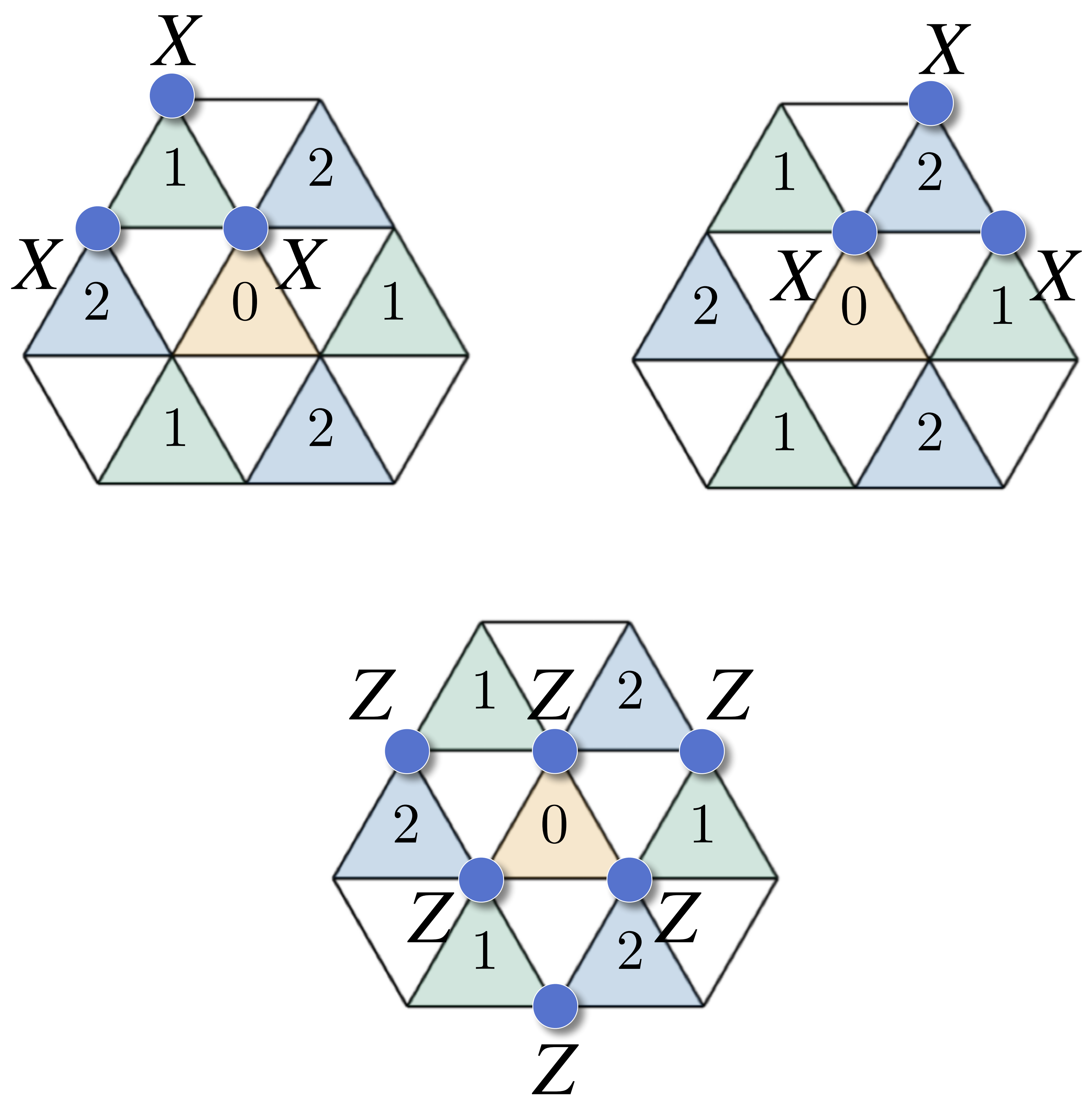

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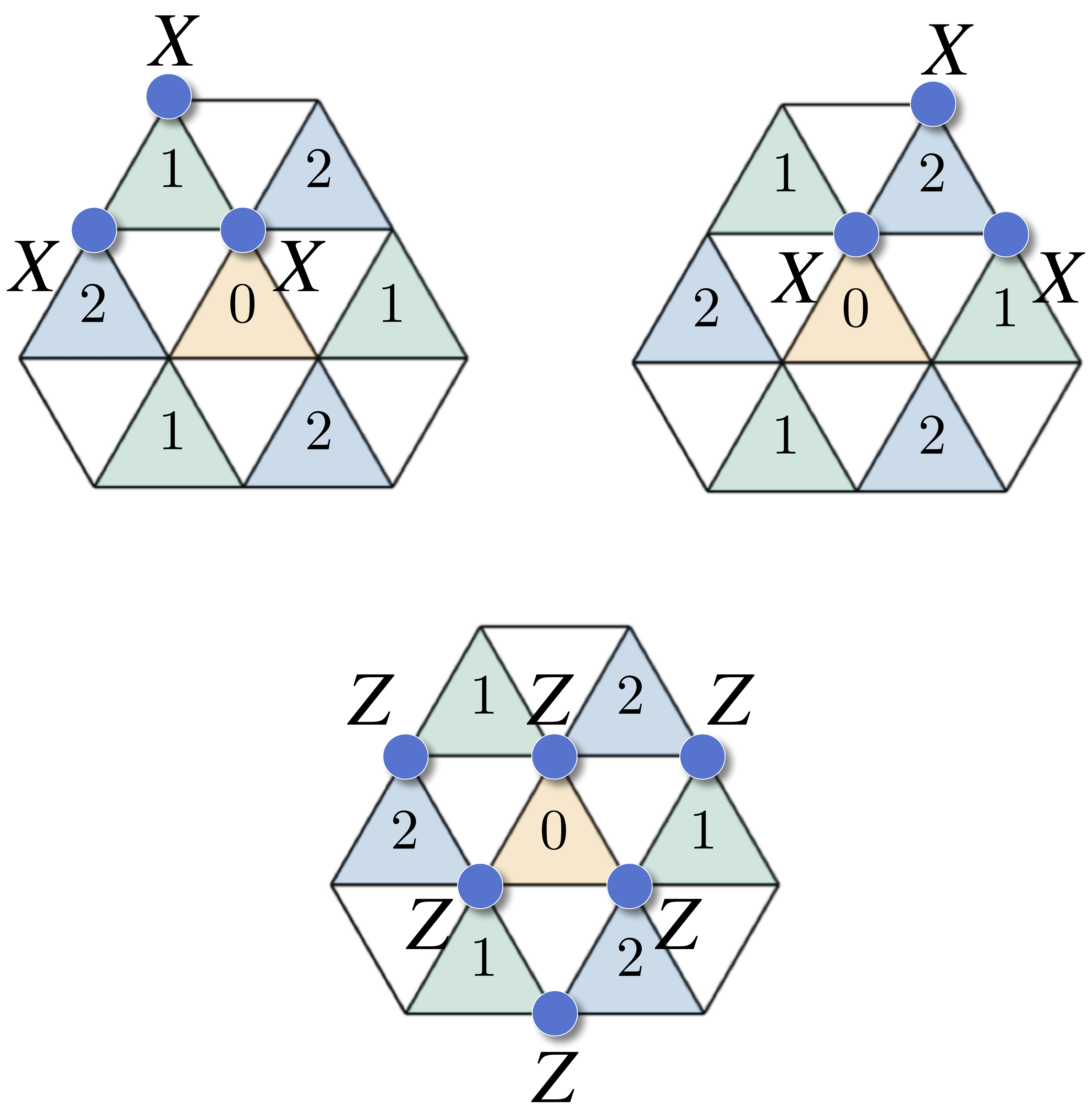

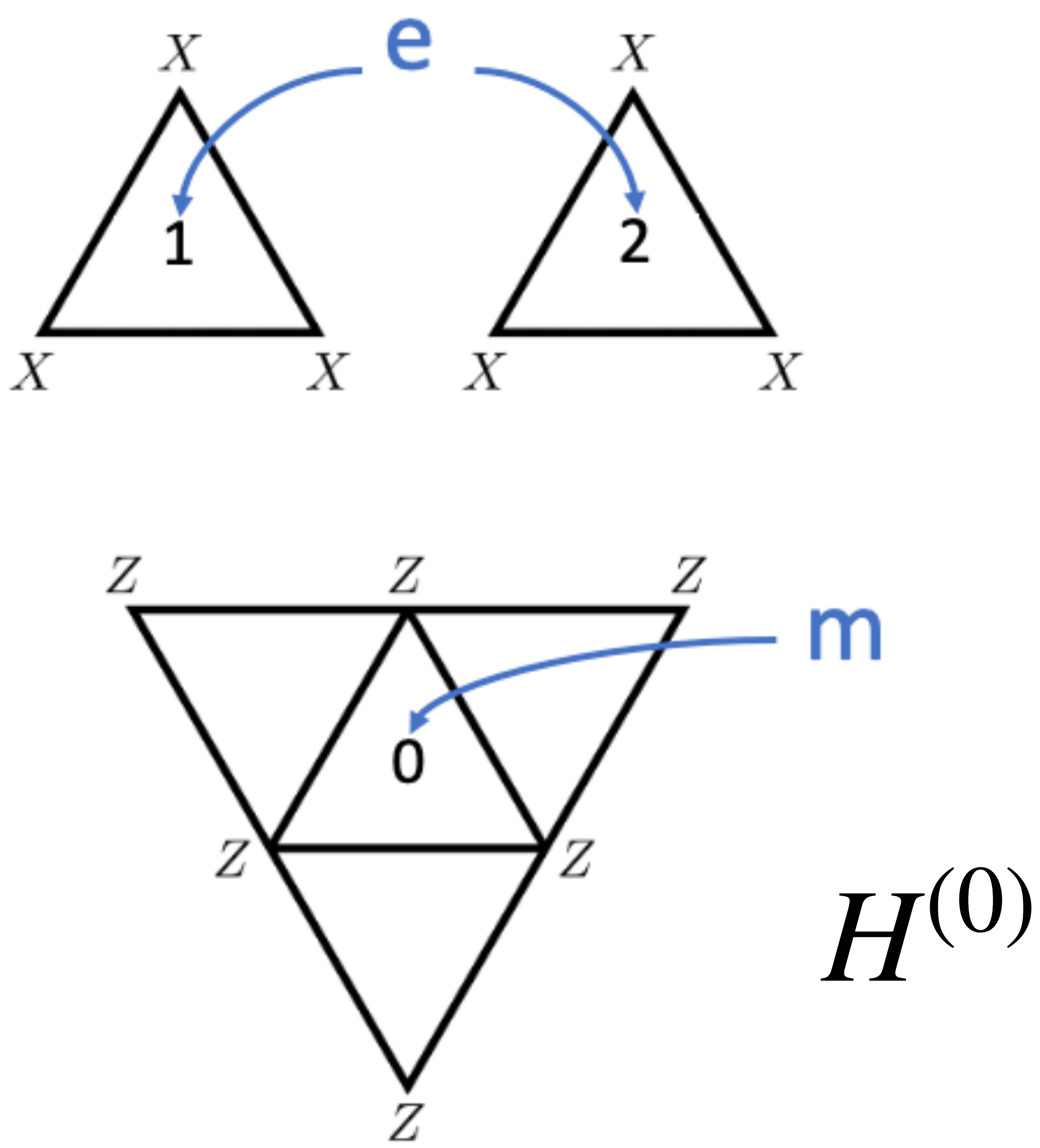

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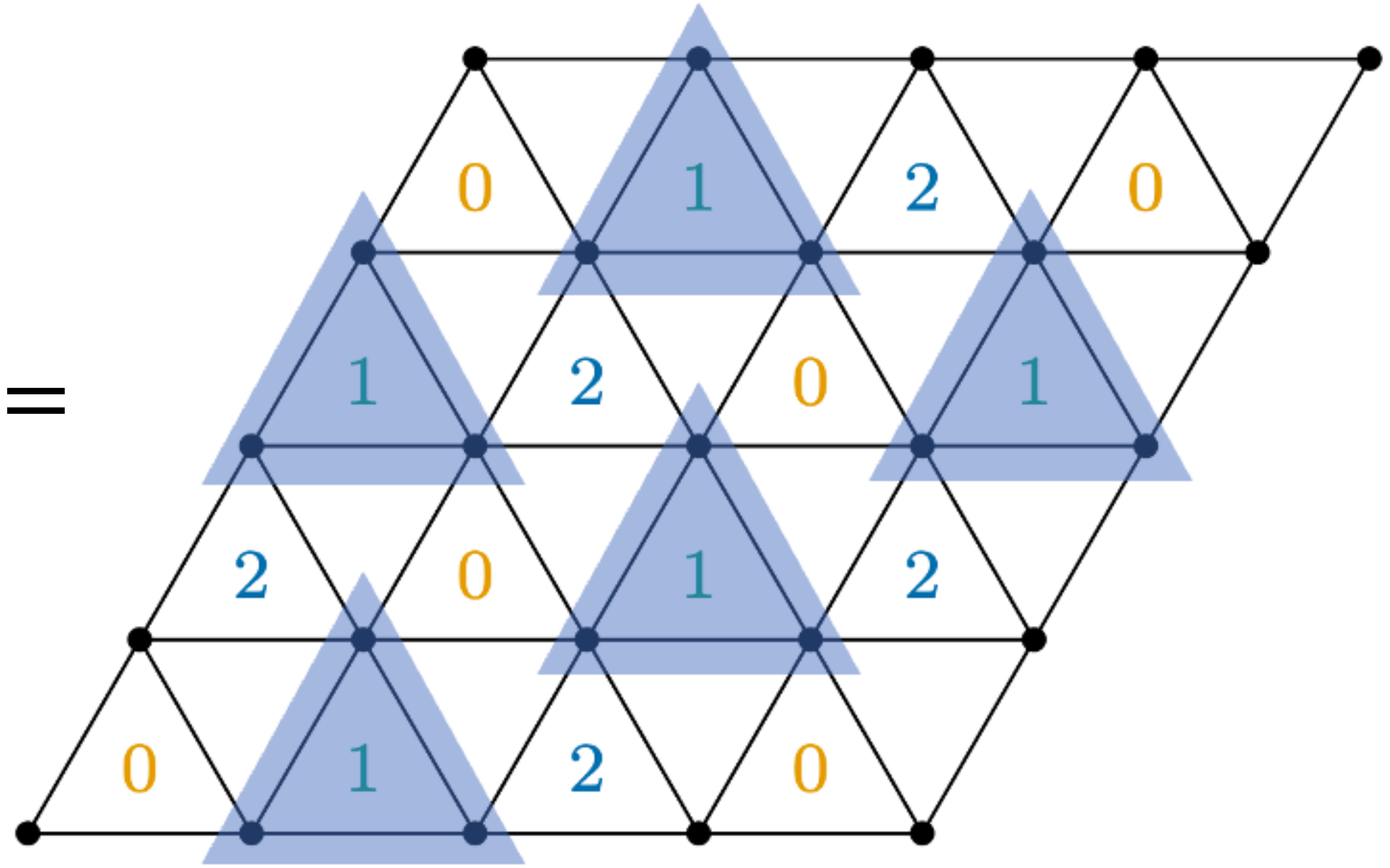
Realizing e - m duality in the toric code model

$\Gamma_1 =$



Sequential **unitary** gauging circuit on a three-qubit chain in counter-clockwise direction.

$$\text{KW}: \; |s_1, s_2, \ldots s_N\rangle \mapsto \sum_{\{t_i\}} \prod_j (-1)^{t_j(s_j + s_{j-1})} |t_1, t_2, \ldots t_N\rangle$$

Kramers-Wannier duality: $X_i \mapsto Z_i Z_{i+1}, \quad Z_i Z_{i+1} \mapsto X_{i+1}\,.$

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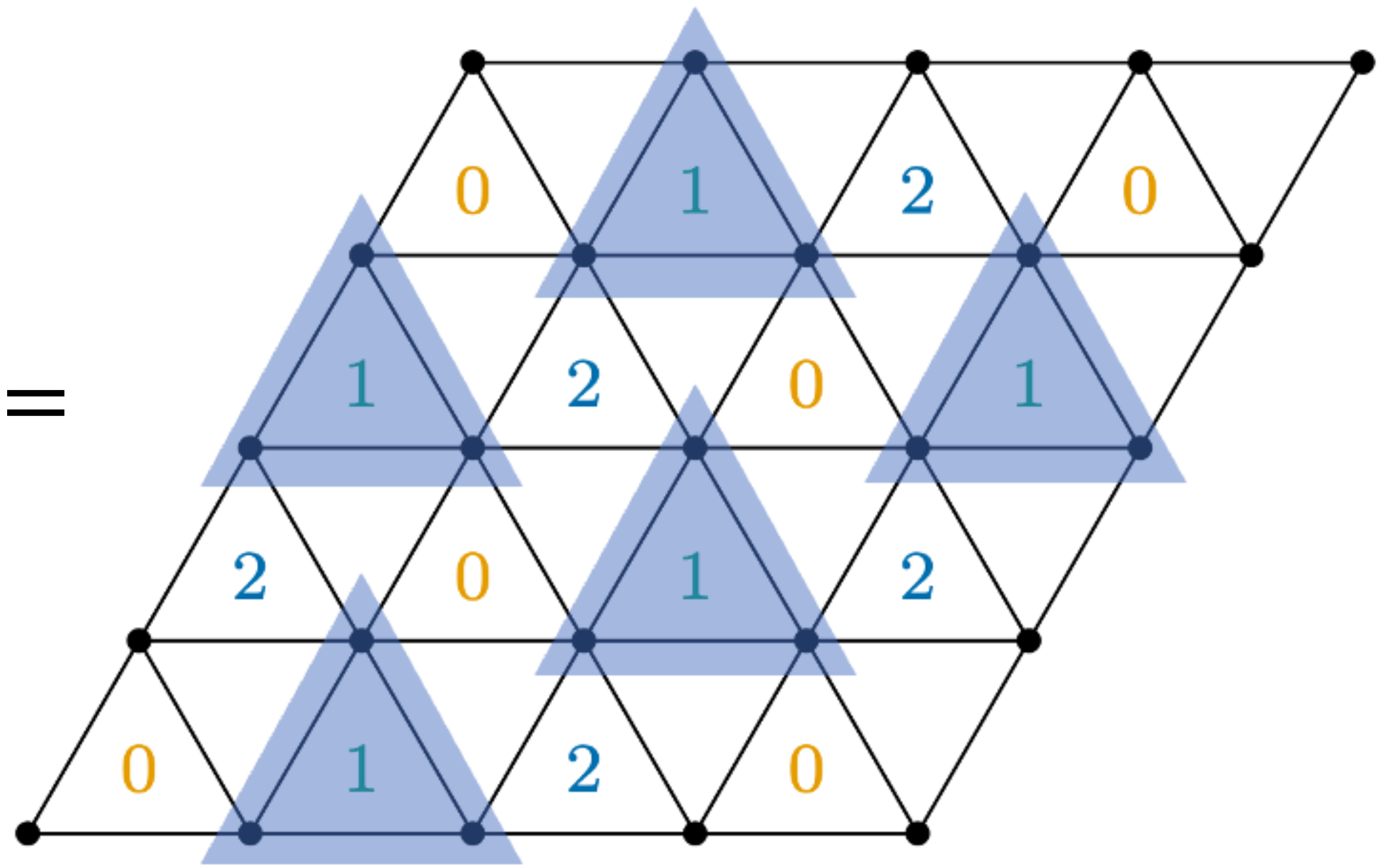
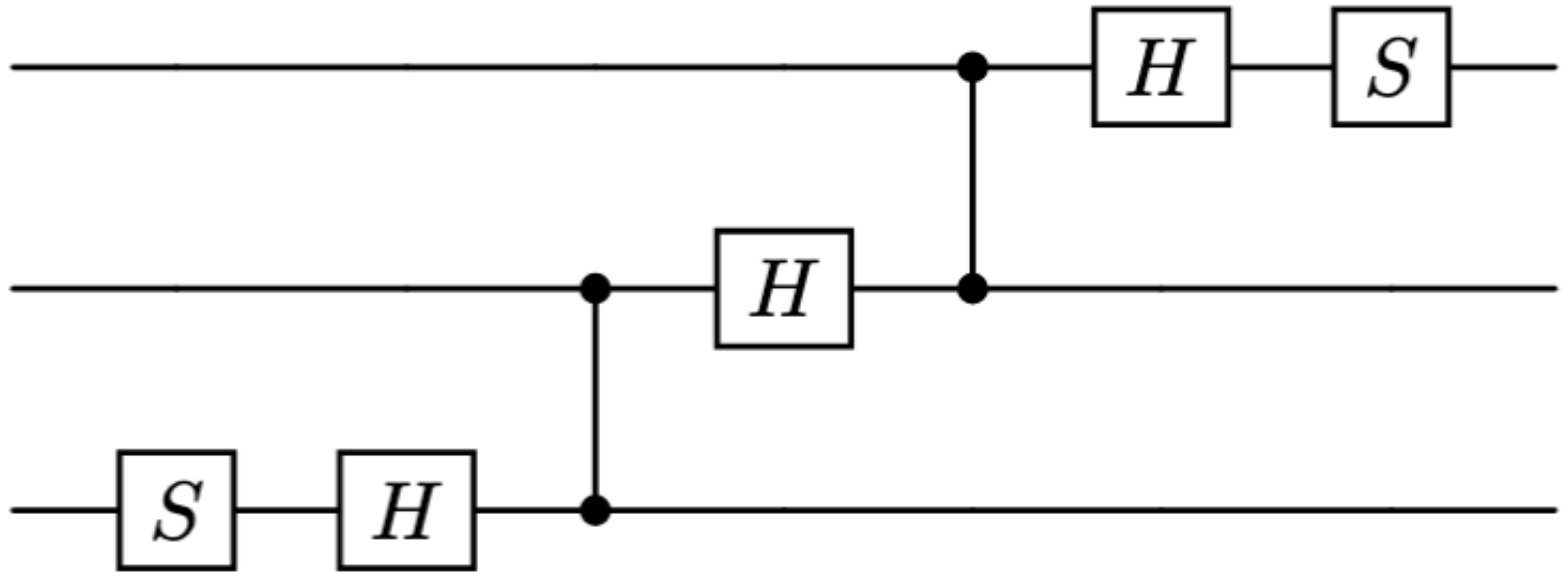
$\Gamma_1 =$ 



$$U_\Gamma = S_1 H_1 CZ_{12} H_2 CZ_{2,3} H_3 S_3$$

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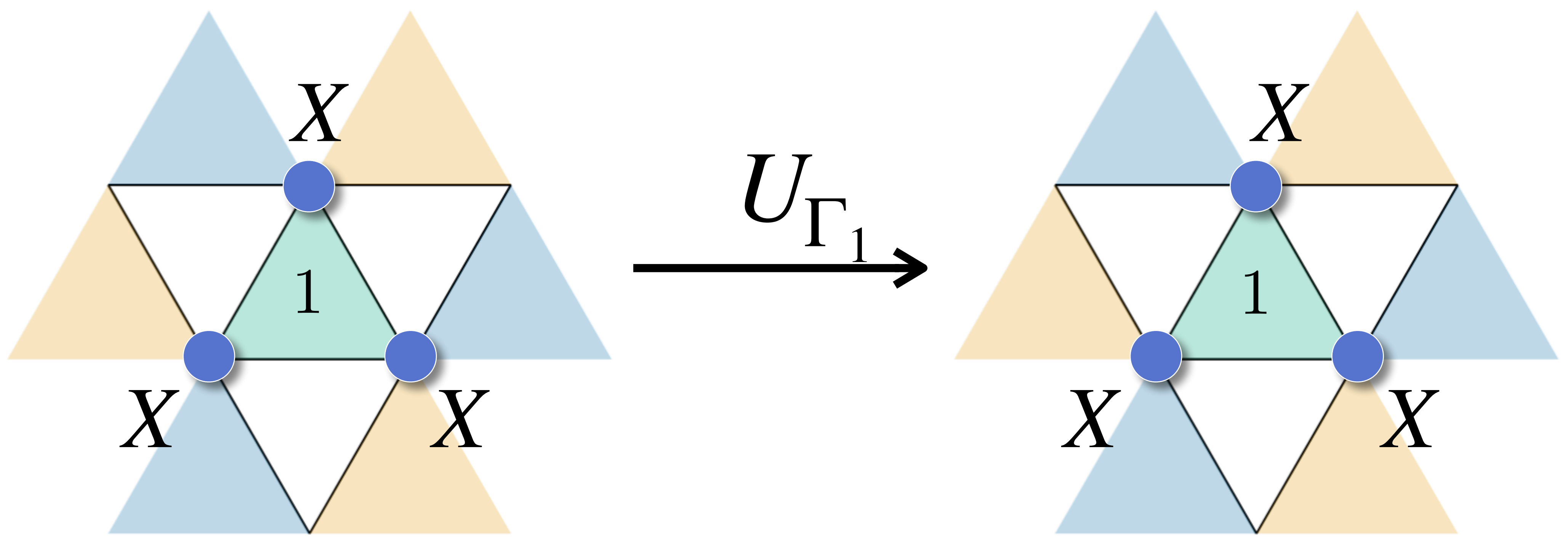

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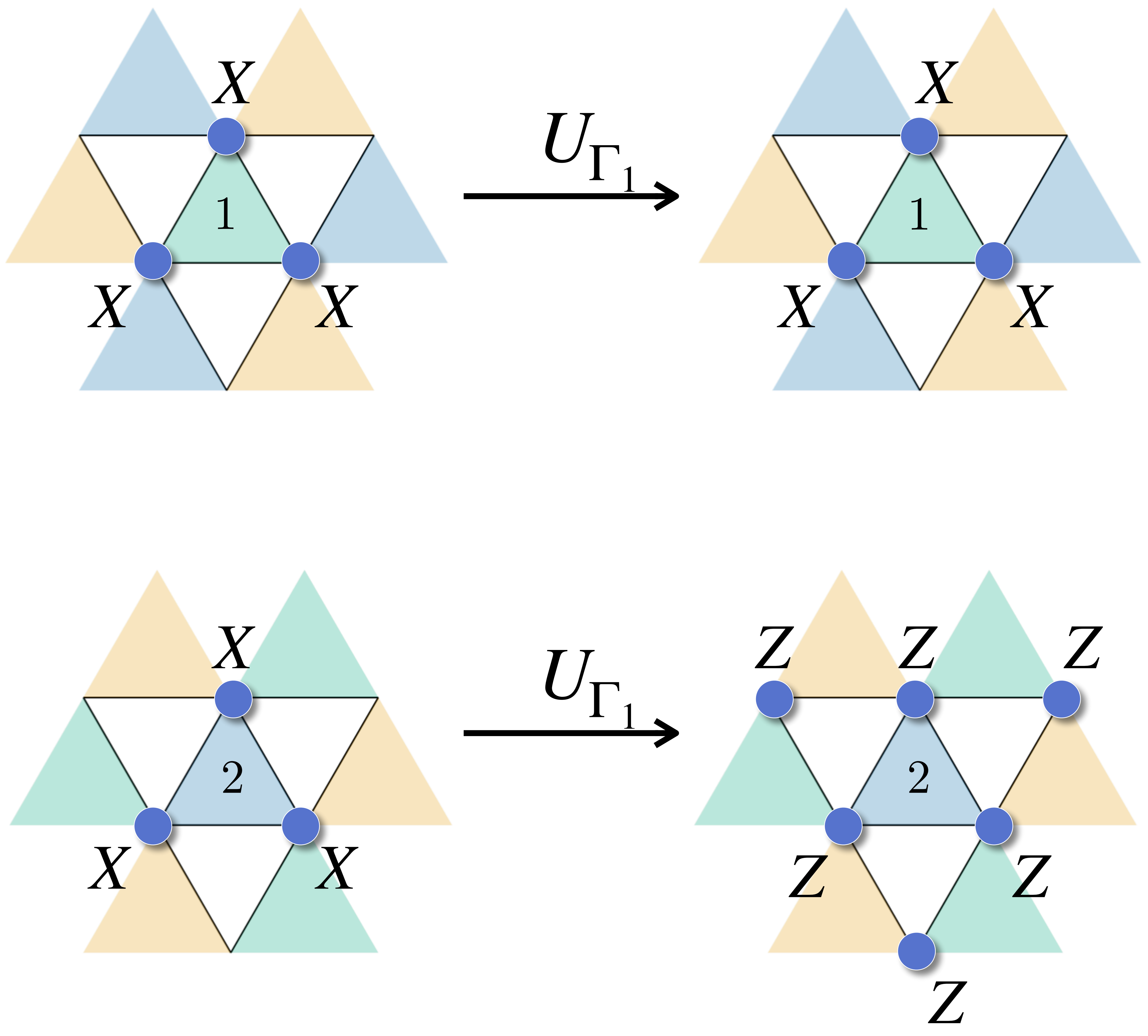

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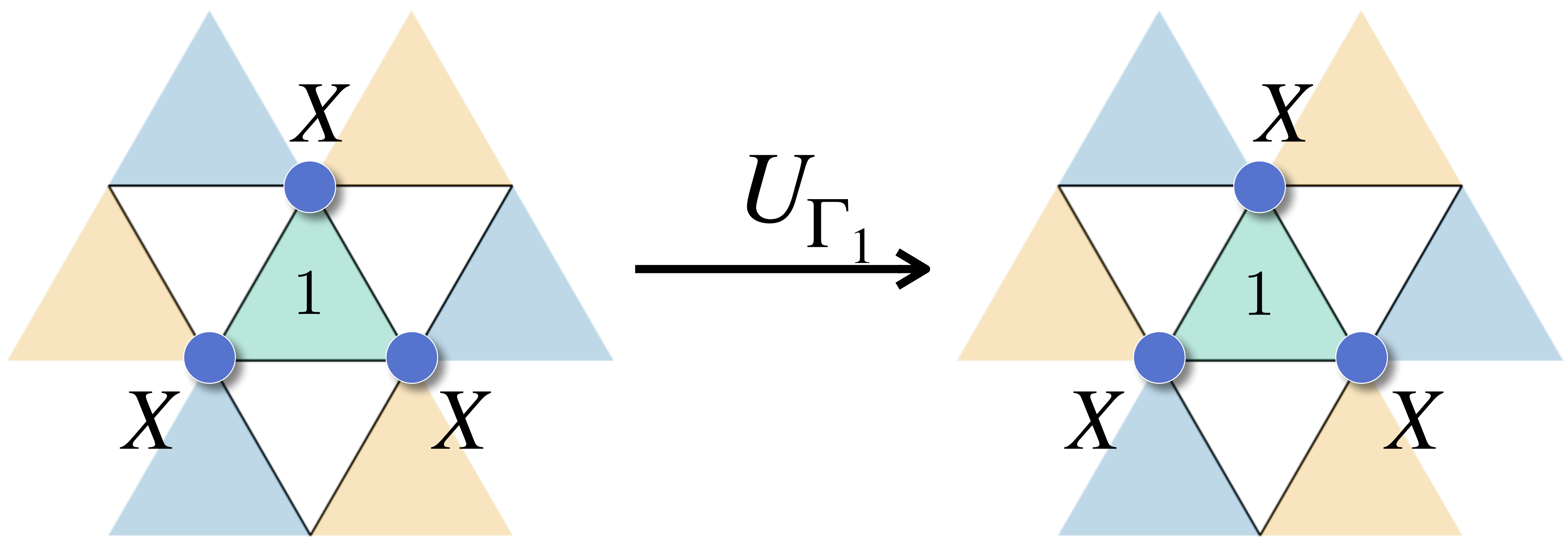

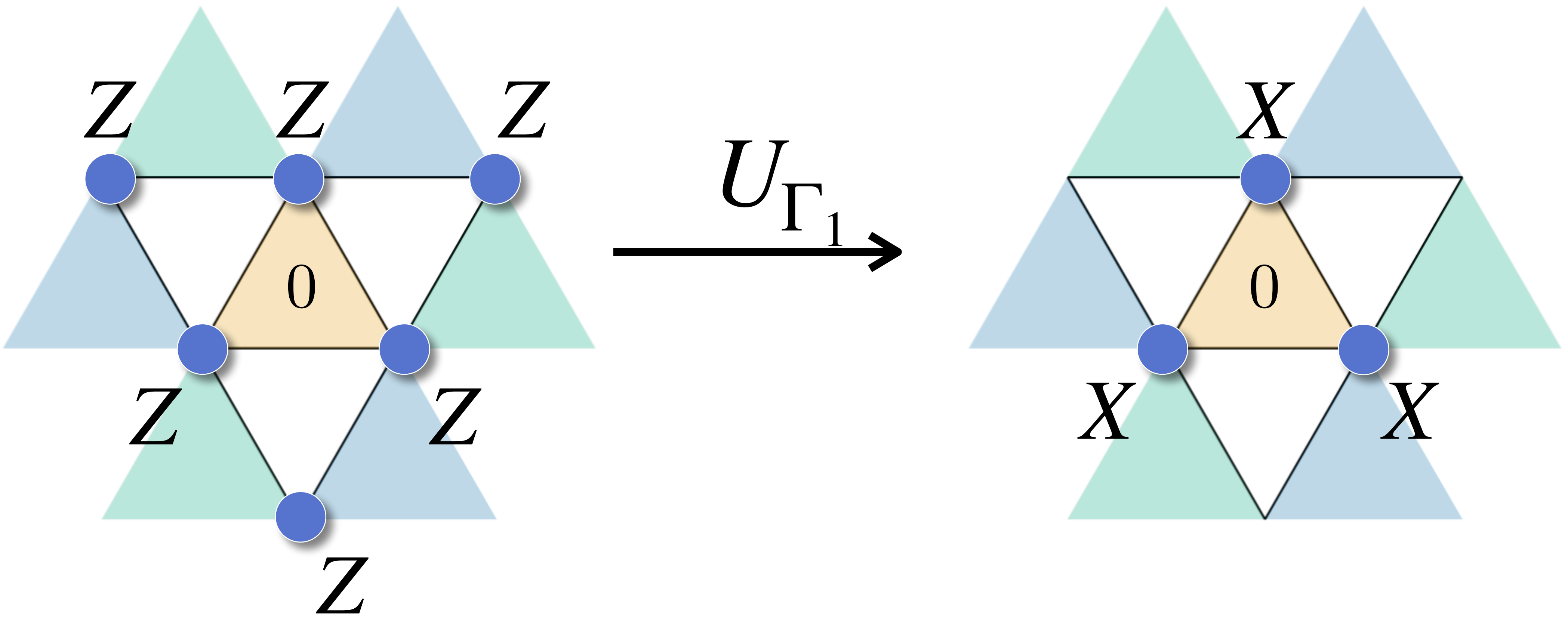

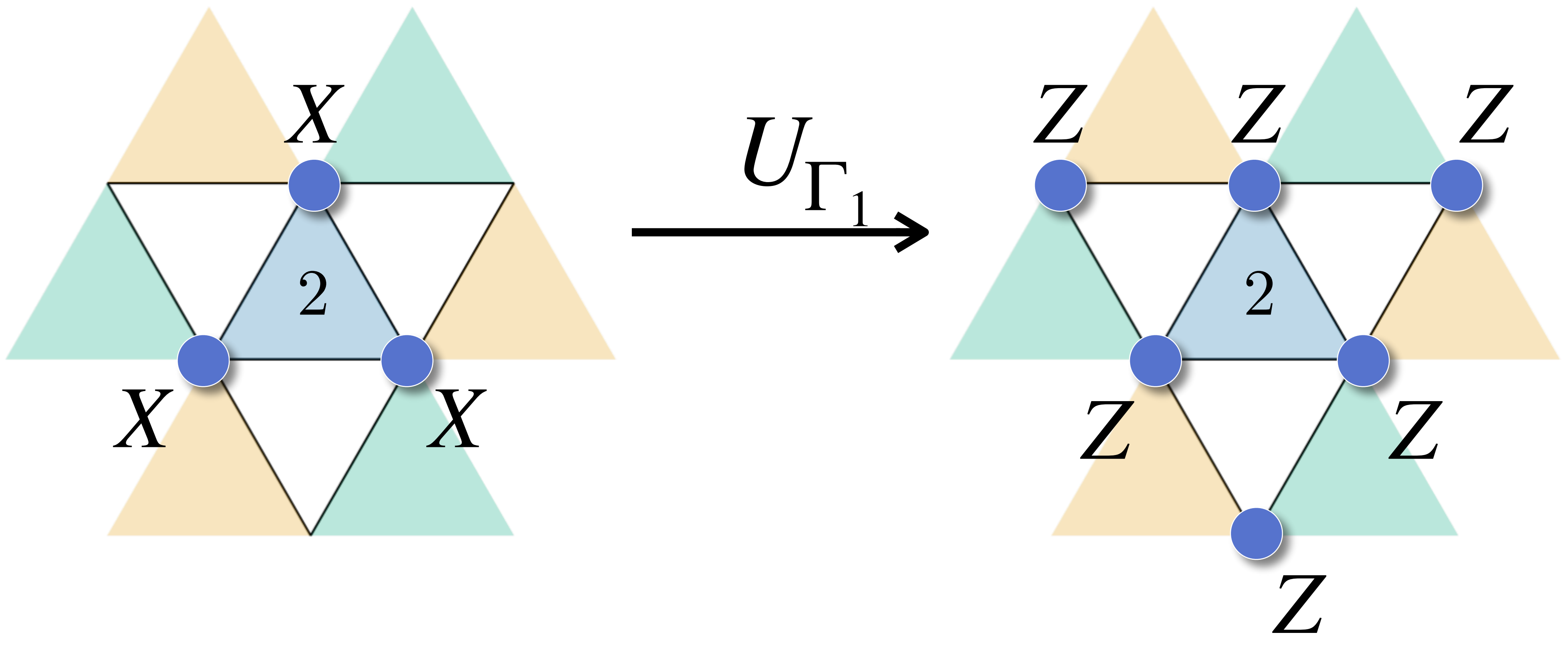

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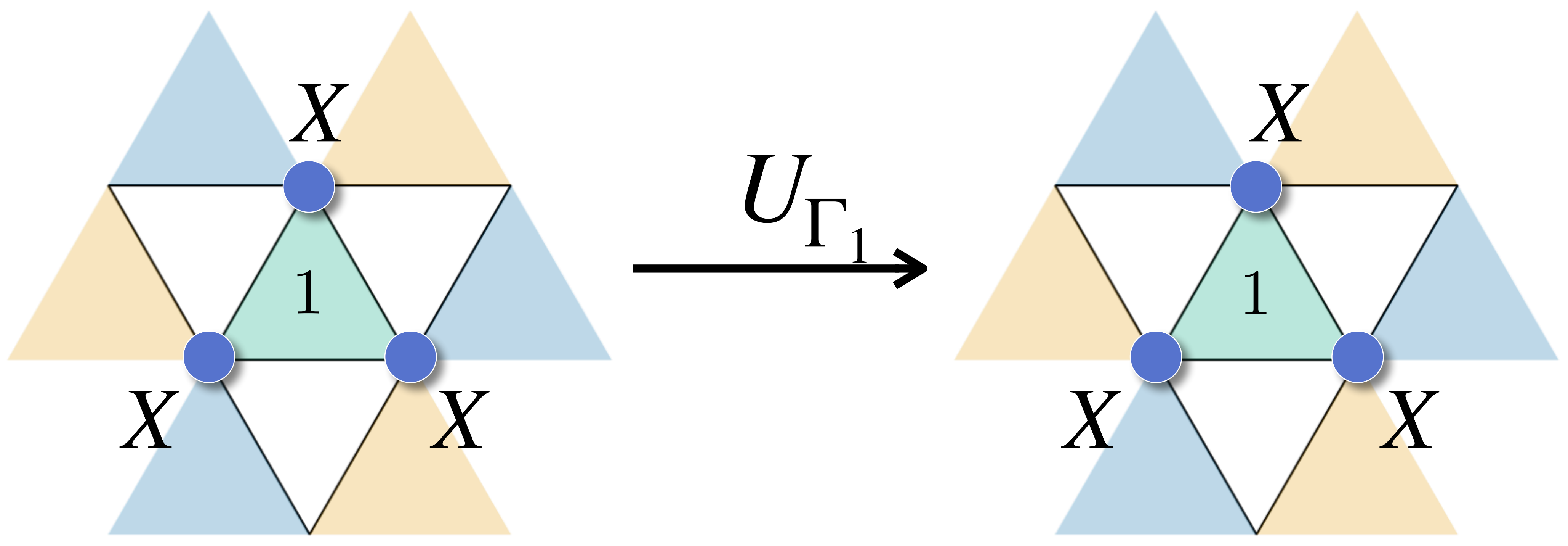

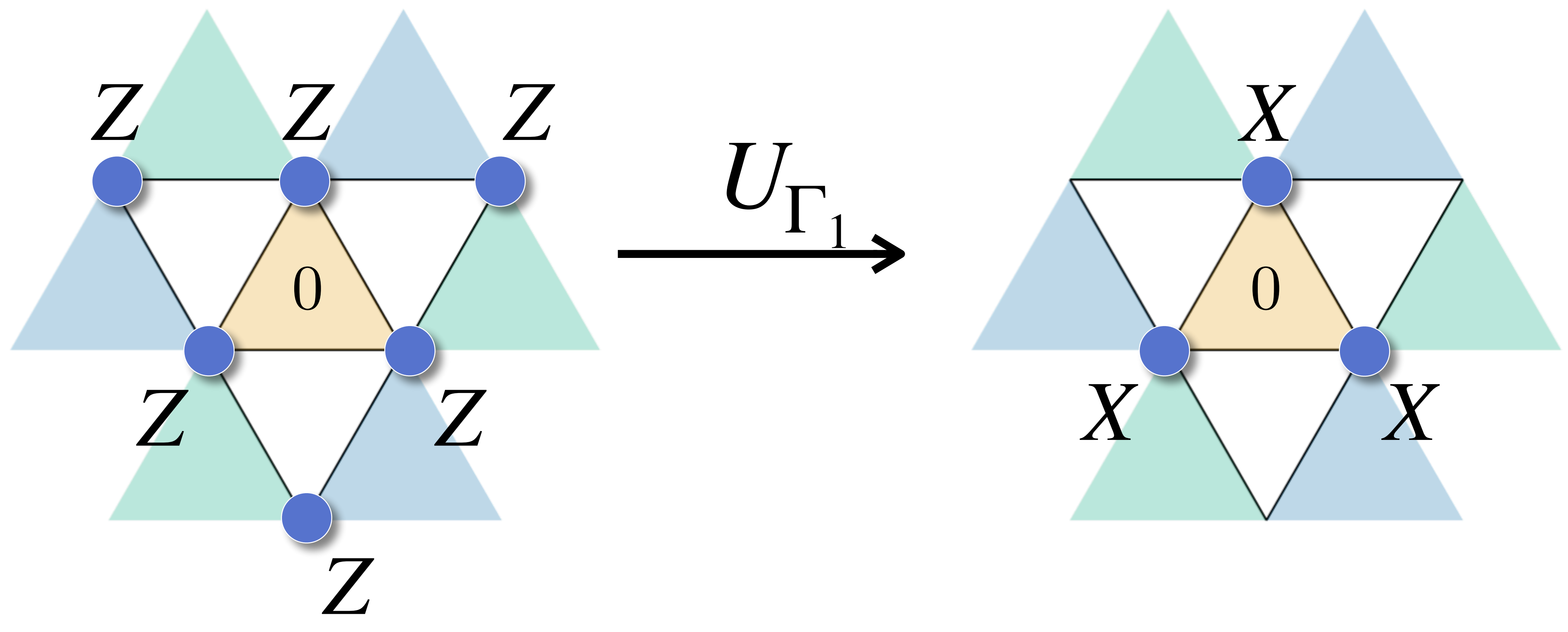

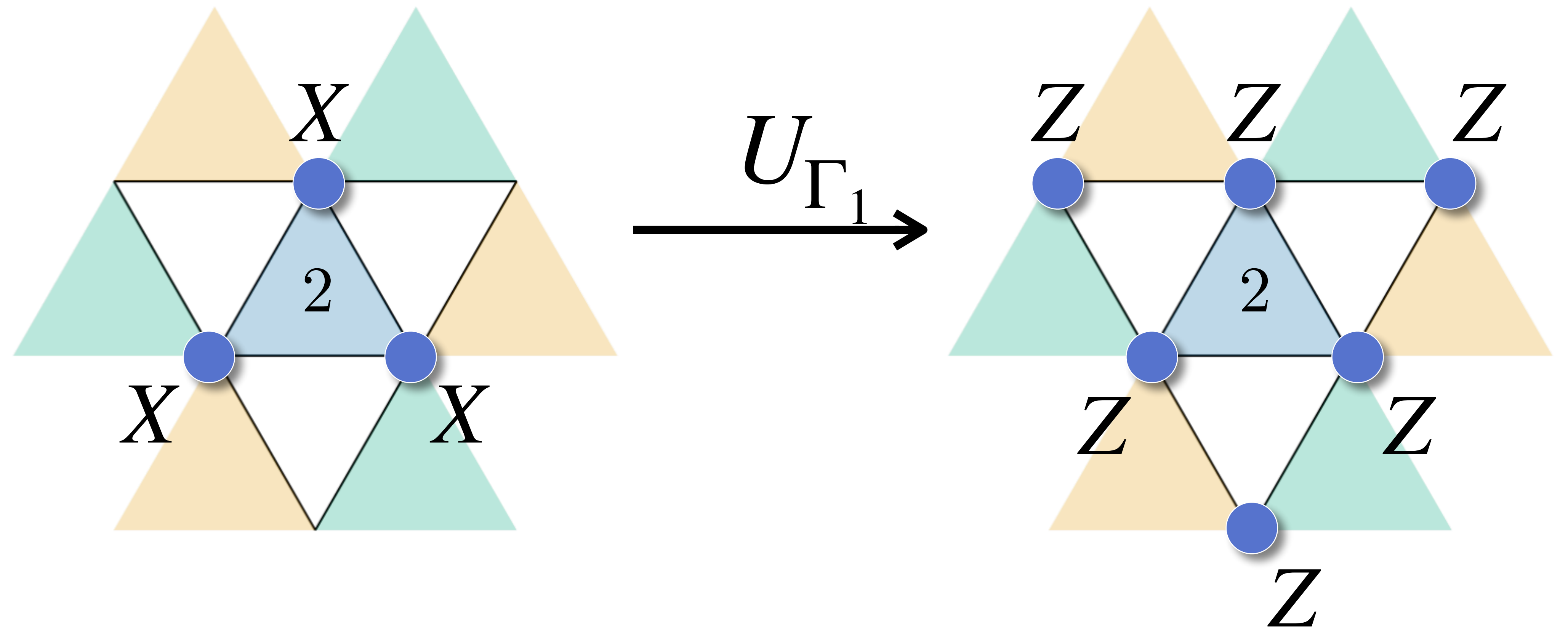

$$H^{(0)} \xrightarrow{U_{\Gamma_1}} H^{(2)}$$

Aasen, Wang, Hastings (2022)

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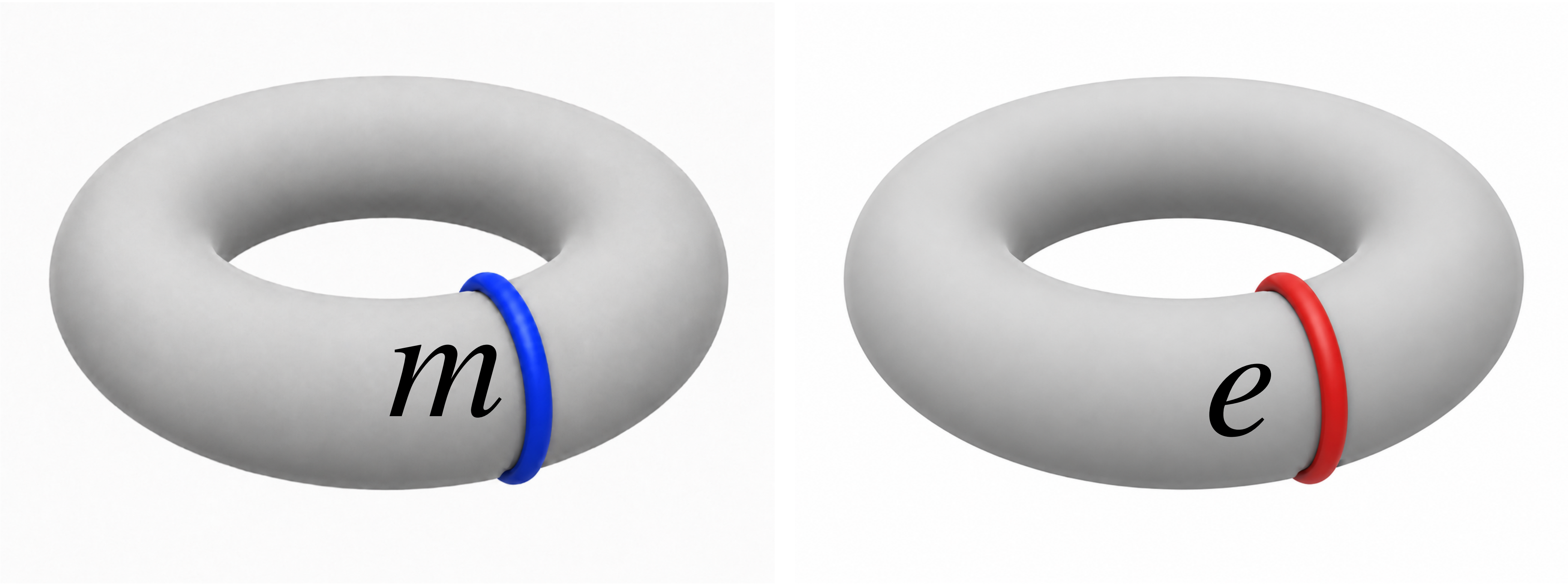

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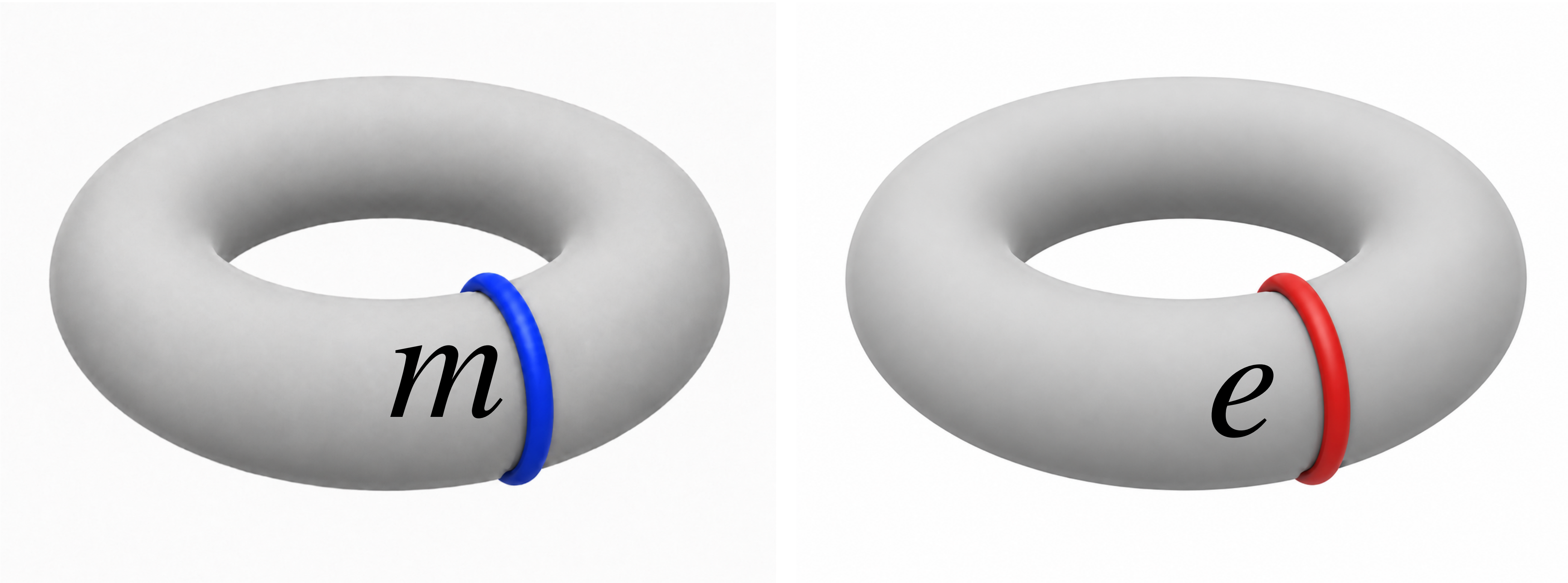

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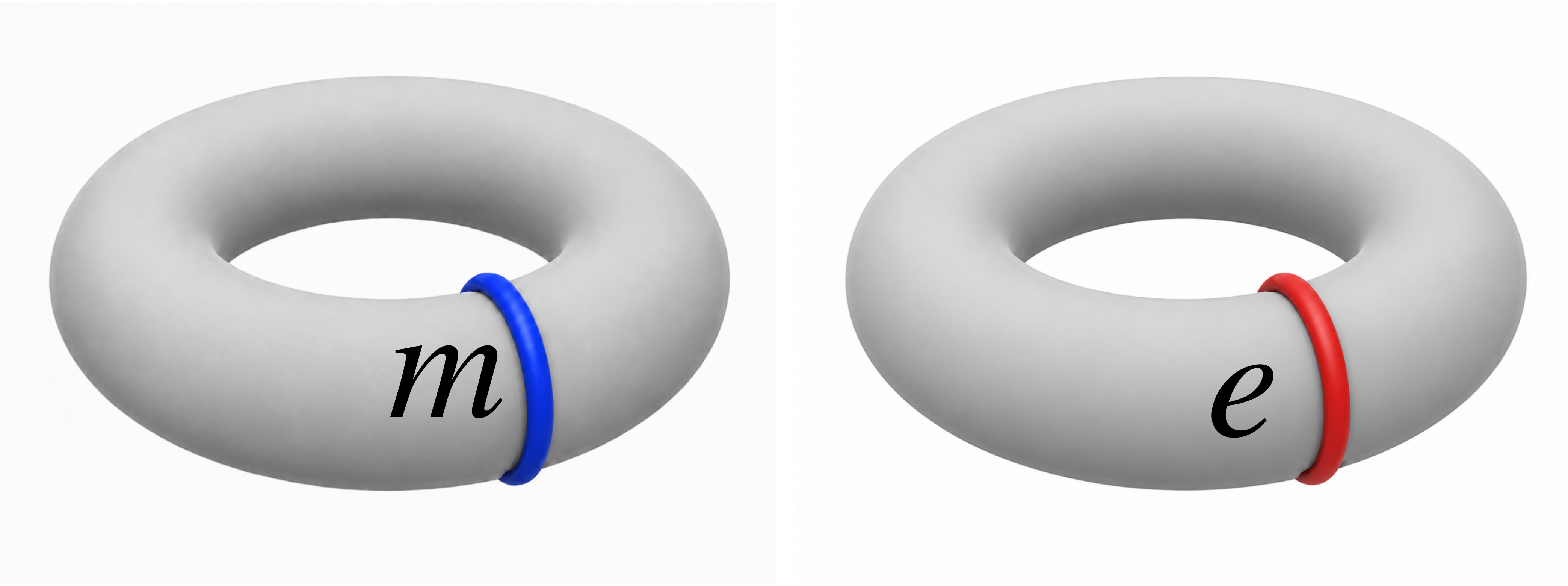
- Realize a logical Hadamard gate with SWAP on a torus.
- The total circuit depth is constant.



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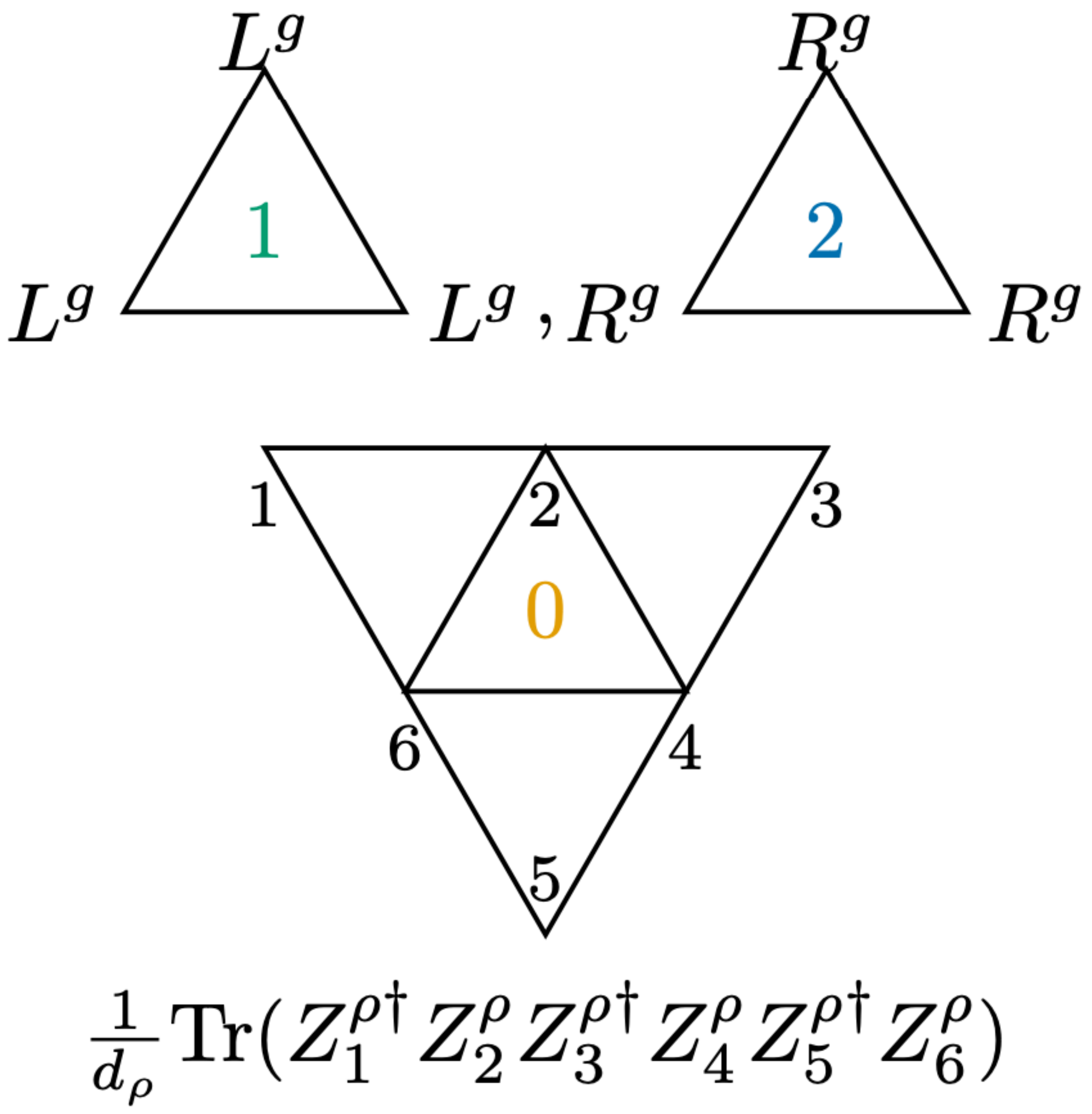
Generalized e - m duality in the quantum double model

- The Hamiltonian of the $D(G)$ quantum double can be defined similarly:



$$L^g = \sum_h |gh\rangle\langle h|\,,\ \ R^g = \sum_h |hg\rangle\langle h|\,,$$

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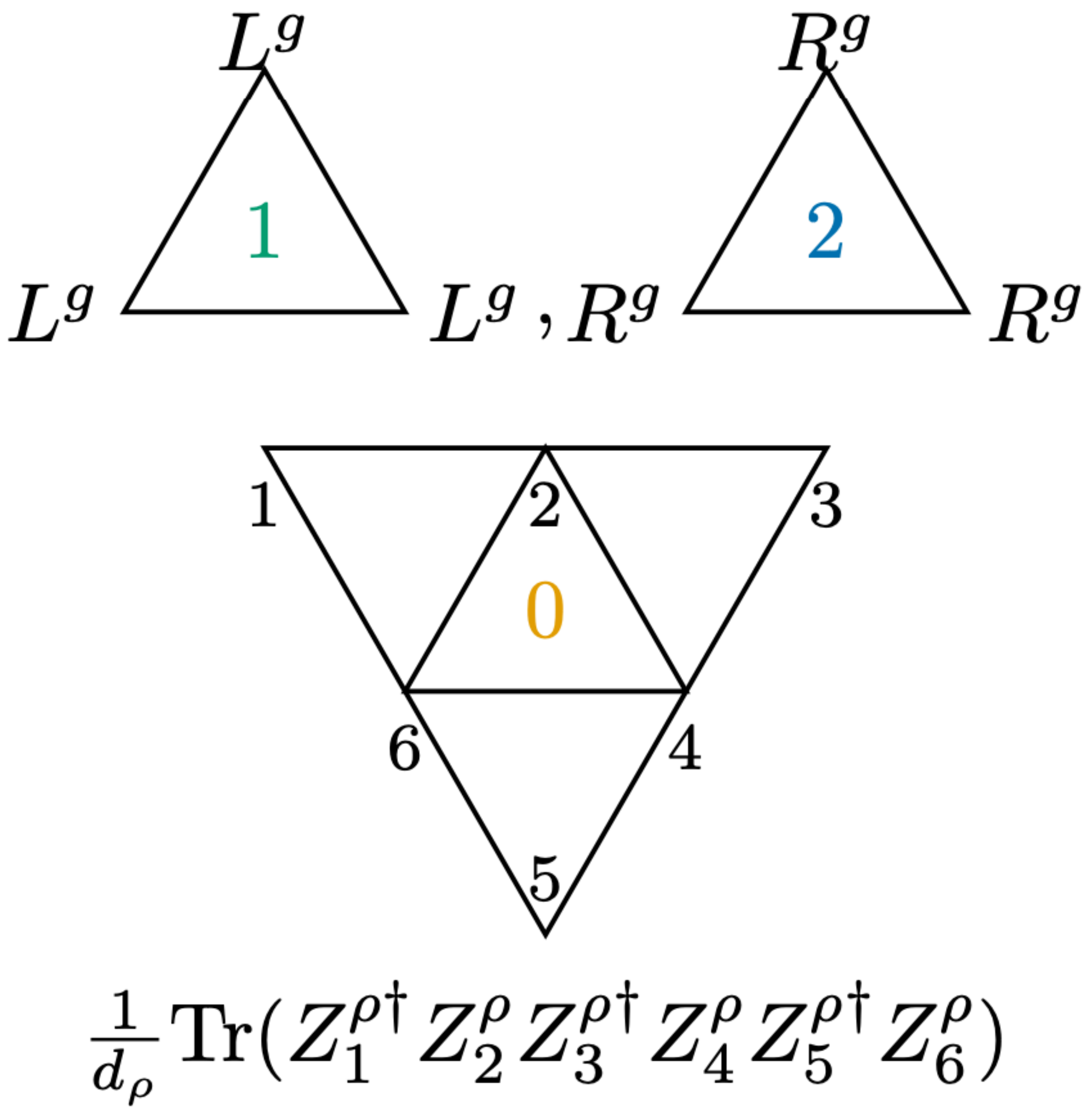

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- The associated Γ are (twisted) gauging some Abelian normal subgroup.

Li, ZS (2026)

Gauging map for G symmetry

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- Consider group extension given by the following exact sequence:

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q = G/N \longrightarrow 1$$

with conjugation action $\sigma : Q \to \text{Aut}(N)$, a 2-cocycle $\omega \in Z^2(Q, N)$.

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- A G element can thus be given by a pair (n, q), with the multiplication rule:

$$(n_1, q_1) \cdot (n_2, q_2) = (n_1 \cdot \sigma^{q_1}(n_2) \cdot \omega(q_1 q_2), q_1 q_2)$$

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- An N gauging map can be defined given a bicharacter $\chi : N \times N \to U(1)$.

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- When ω is cohomologically trivial, and χ is invariant under conjugation, we have the following (untwisted) gauging map:

$$\text{KW}: \ |\{n_i, q_i\}\rangle \mapsto \sum_{\{t_i\}} \prod_j \chi(m_j, \bar{n}_{j-1} n_i) \, |\{m_i, q_i\}\rangle$$

Generalized e - m duality in the quantum double model

$$H^{(0)} \xrightarrow{\Gamma_1} H^{(2)} \xrightarrow{CC \circ \Gamma_0 \circ CC^\dagger} H^{(1)} \xrightarrow{\Gamma_2} H^{(0)}$$

Li, ZS (2026)

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- The above cycle can also be realized in ***constant depth***, with local Hadamard, CZ, and controlled-conjugation gate.
- Ex: The $C \leftrightarrow F$ anyon permutation of the $S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ quantum double can be realized by gauging the $\mathbb{Z}_3$ subgroup.

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General self-dual twisted gauging map

Li, ZS (2026)

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Li, ZS (2026)

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    - χ is invariant under conjugation.

Li, ZS (2026)

General self-dual twisted gauging map

- In the most general case, the following consistency relations must be satisfied:
    - χ is invariant under conjugation.
    - $[\alpha] \in H^2(N, U(1))$ is preserved under conjugation.

Li, ZS (2026)

General self-dual twisted gauging map

- In the most general case, the following consistency relations must be satisfied:
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- The general twisted gauging map exists when the above consistency conditions are simultaneously satisfied.

Li, ZS (2026)

Mathematical structure

- Mathematically, we have the following relation:

$$\mathscr{G}(G) \cong \text{ Anyon permutations in } D(G)$$

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Can be realized (almost) transversally!

Warman, Schäfer-Nameki (2025)

Kobayashi, Zhu, Hsin (2025)

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Kesselring et al. (2018)
Davydova et al. (2024)

The order of the anyon permutation group is 72. $\mathcal{G}(\mathbb{Z}_2^2) = \mathbb{Z}_2 \ltimes (S_3 \times S_3)$

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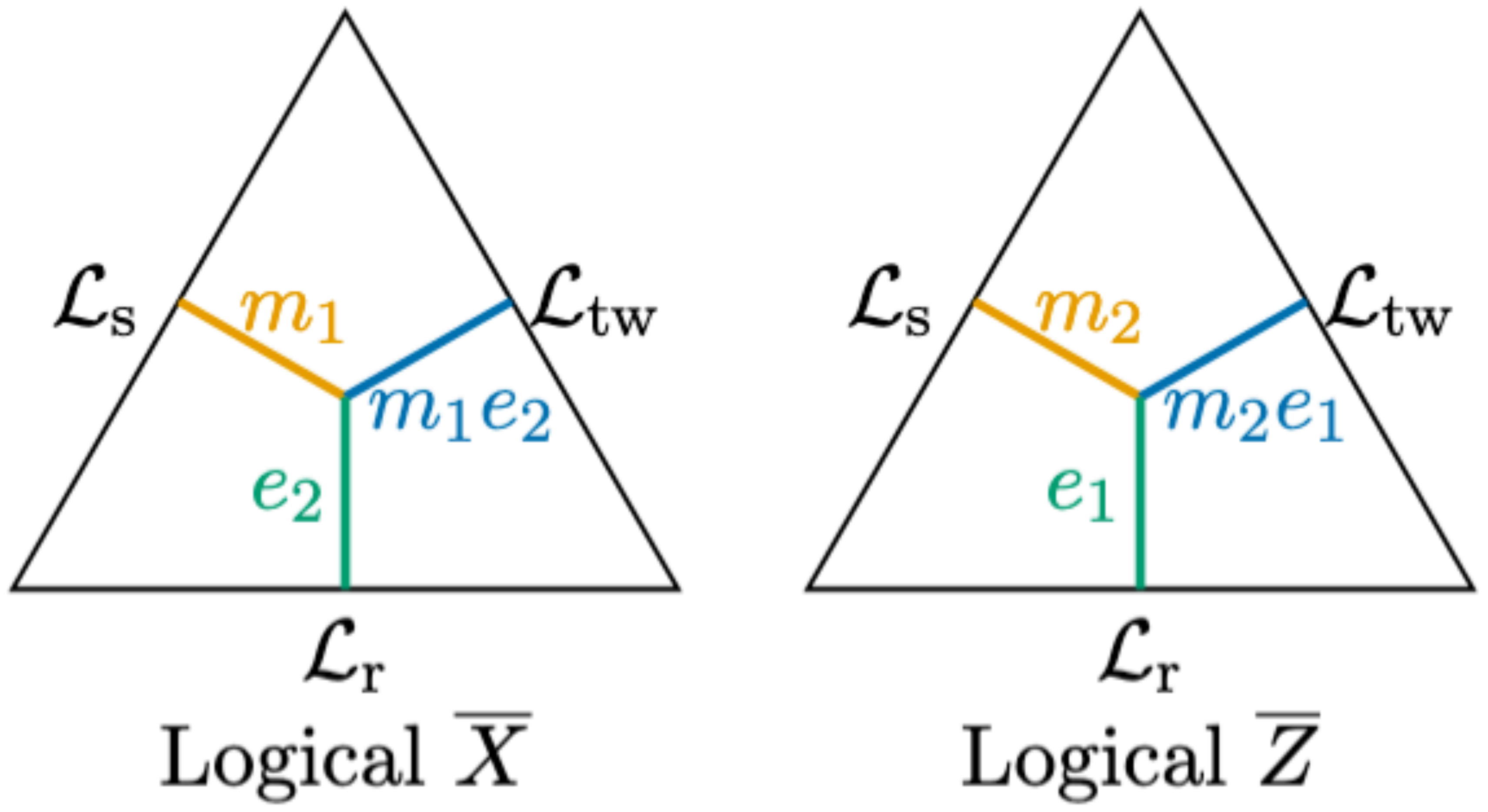

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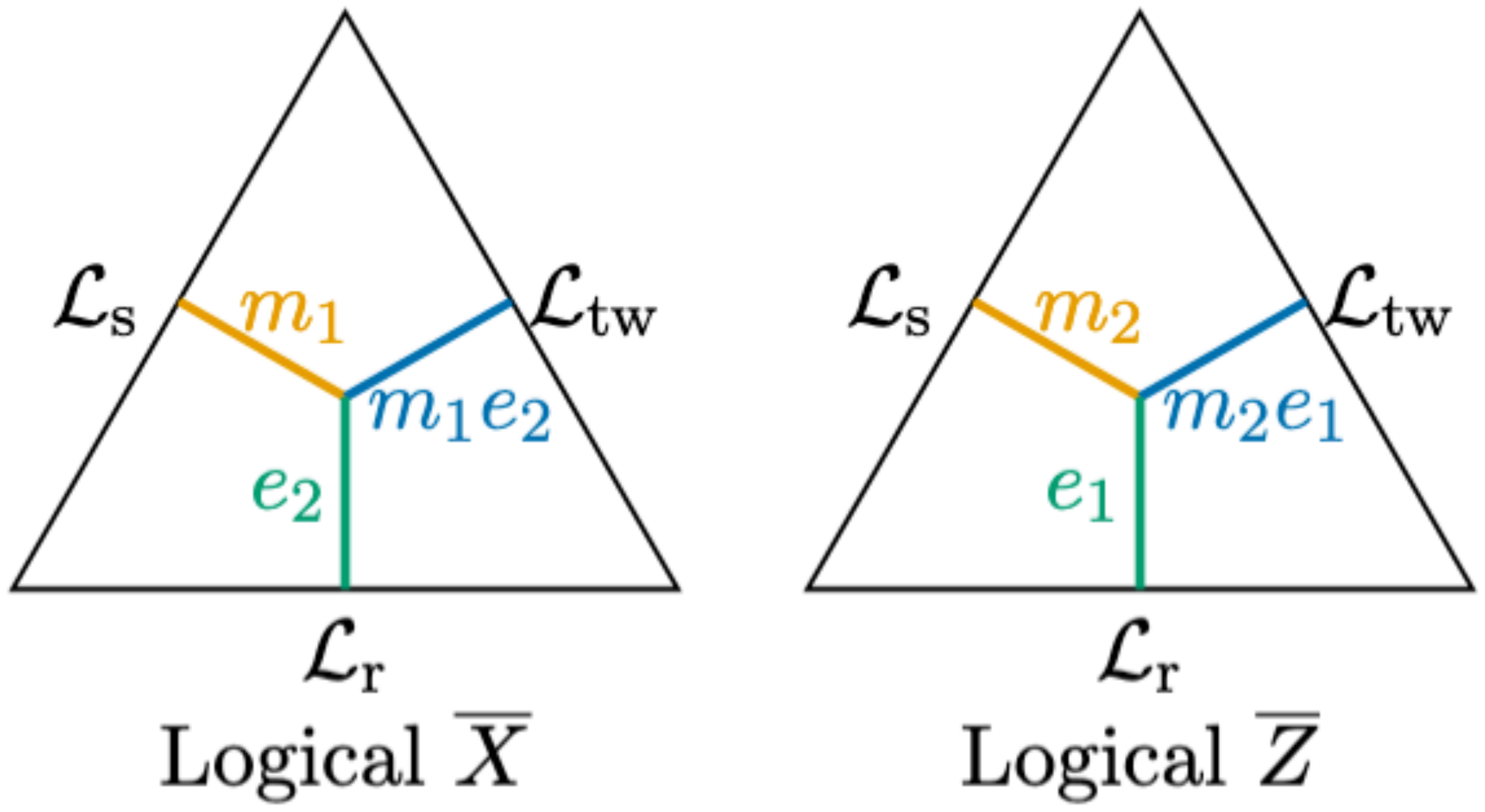

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Smooth Boundary

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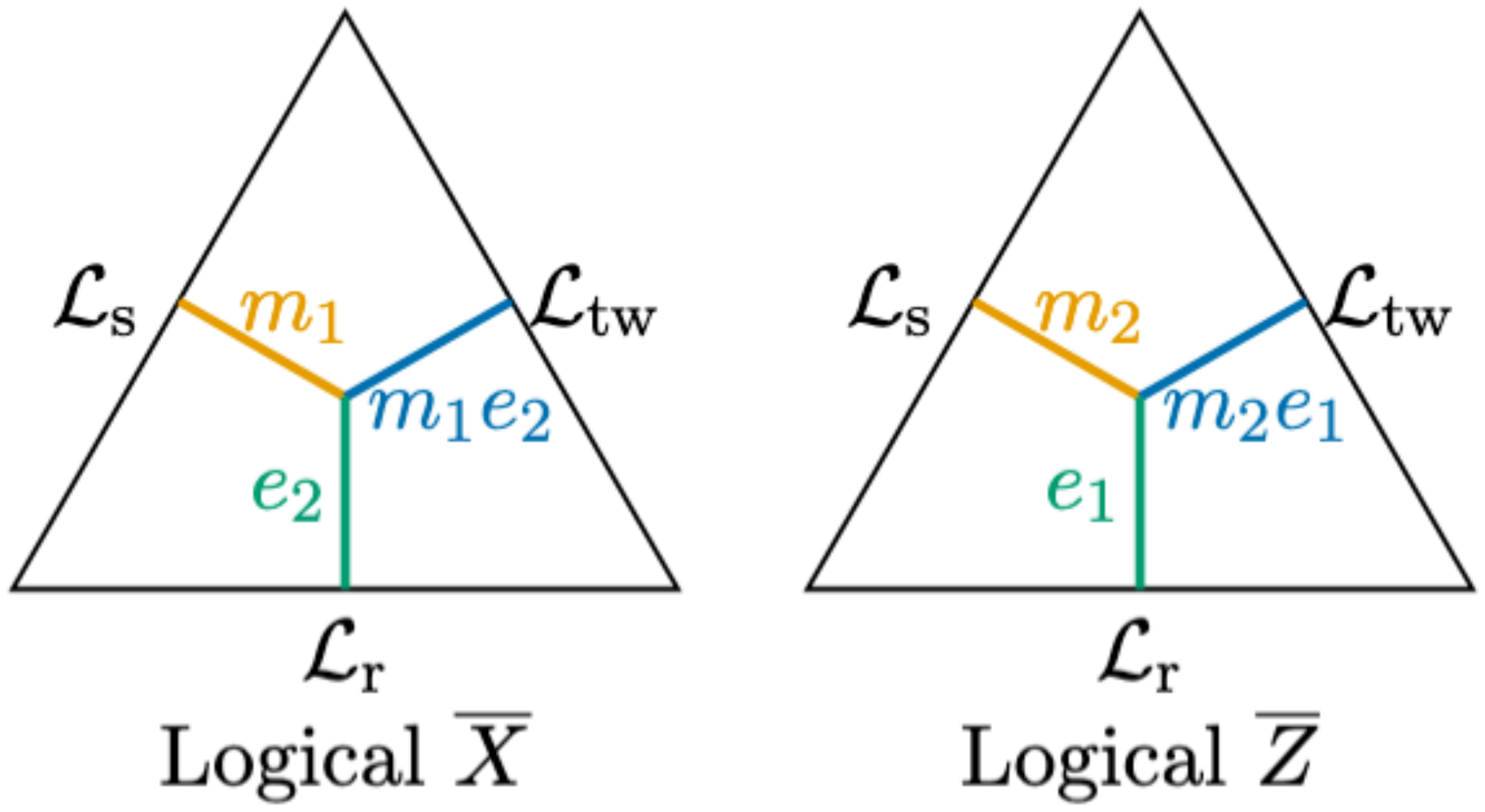

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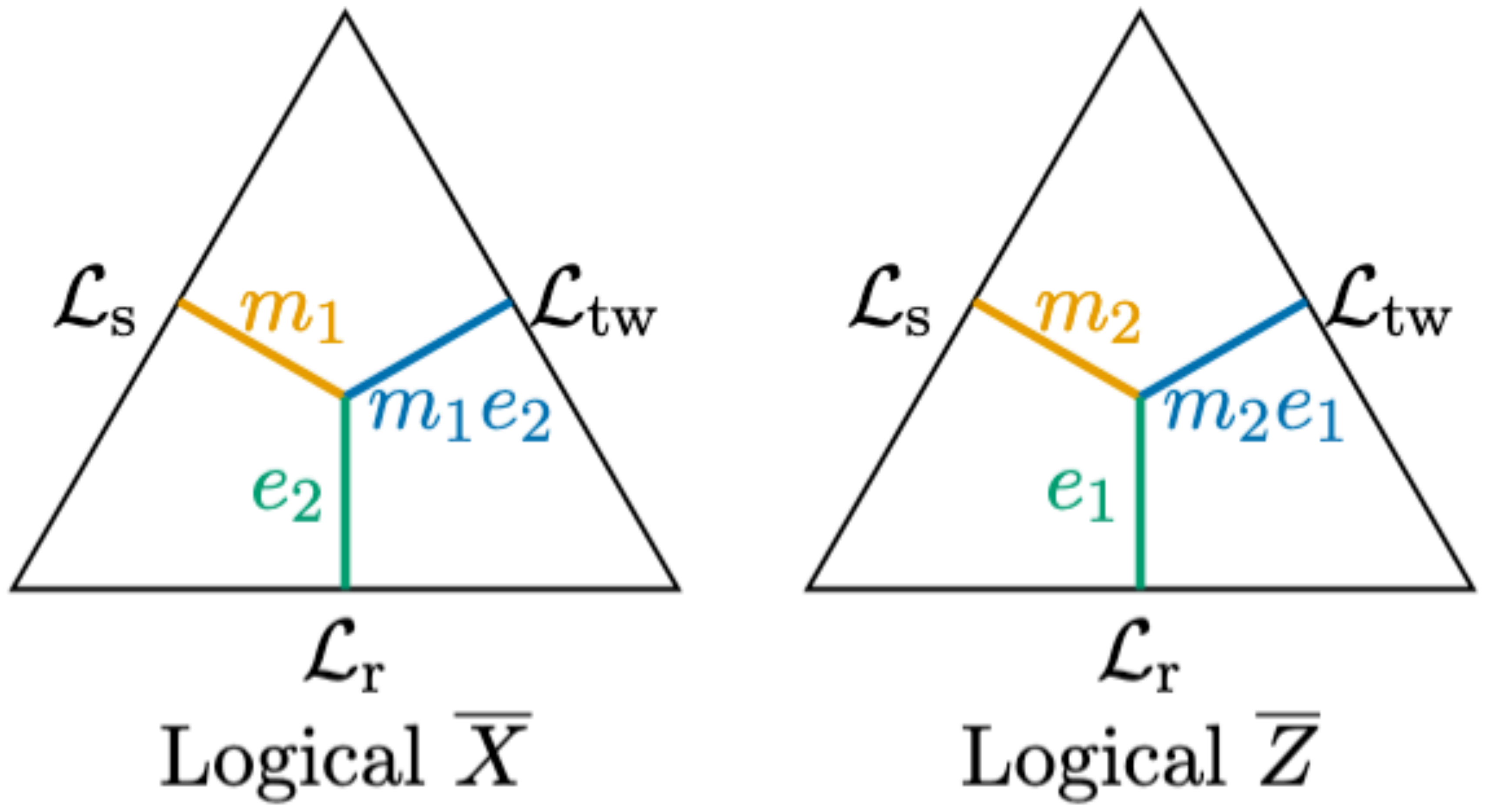

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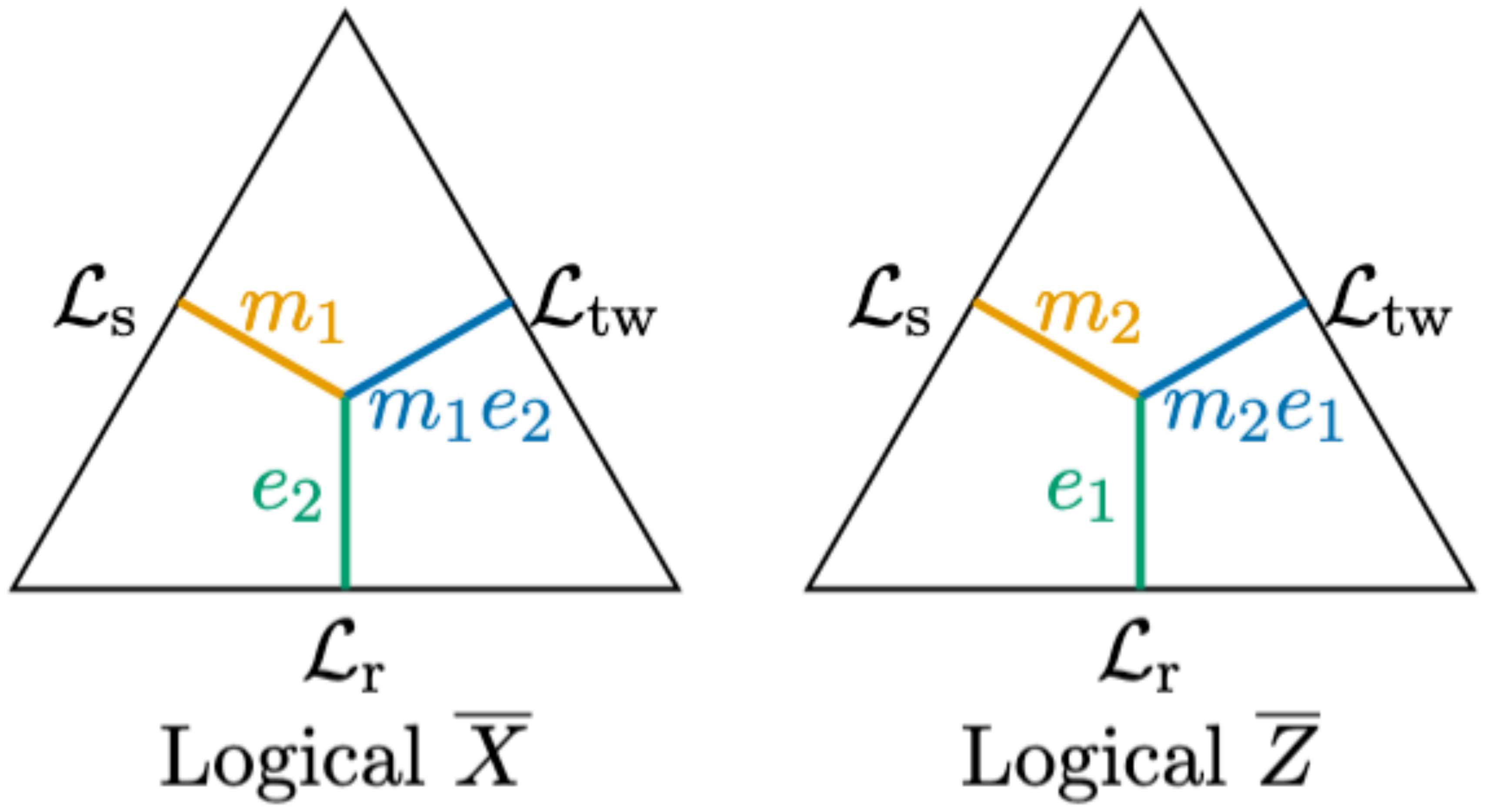

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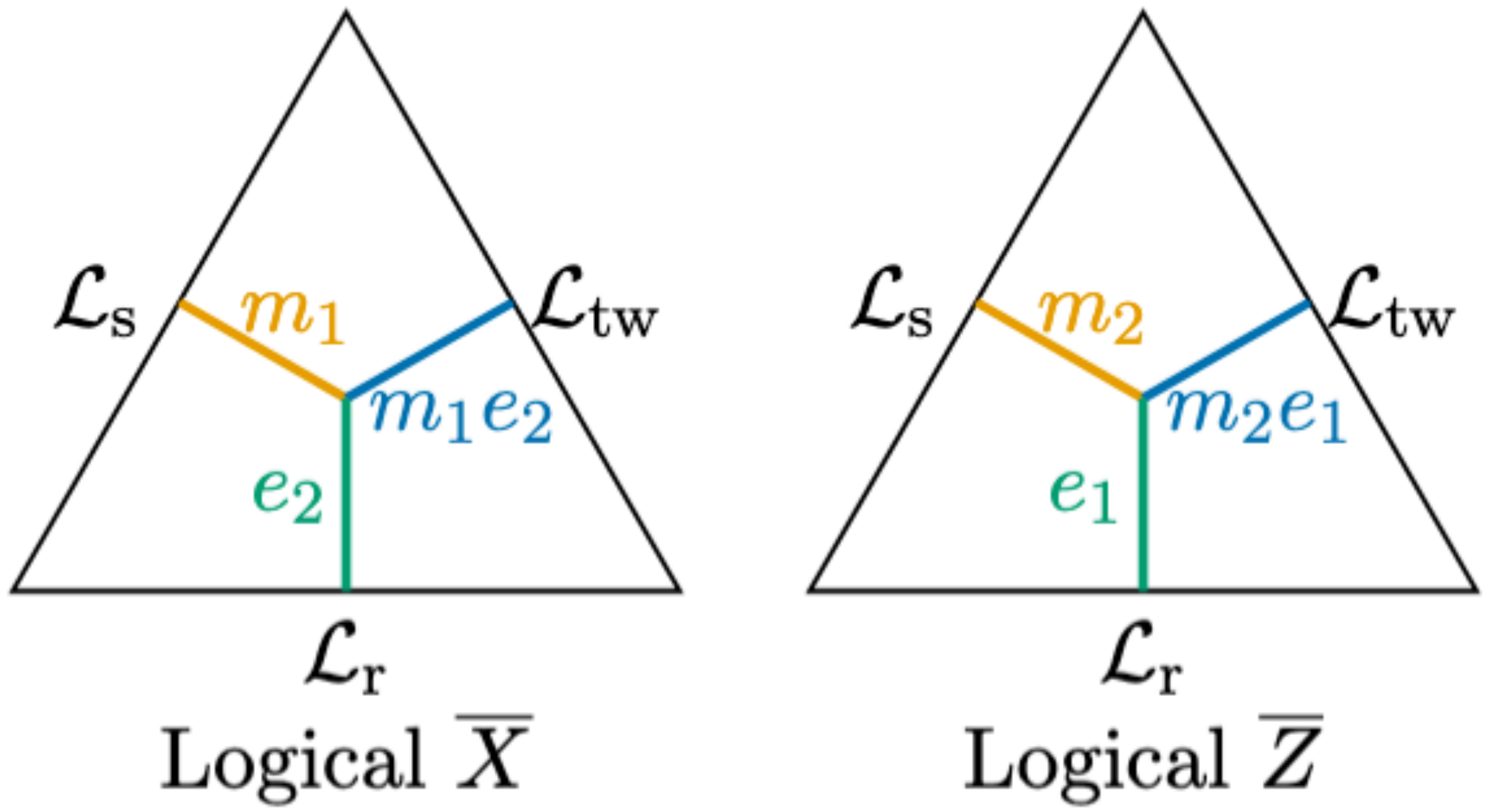

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We now have all the (almost) transversal single qubit Clifford gates.

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Generalized gauging — Kobayashi, Zhu, Hsin (2025)

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Gauged SPT stacking — Warman, Schäfer-Nameki (2025)

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$$m_g \mapsto m_r, \quad e_g \mapsto e_r, \quad m_r \mapsto m_g,$$
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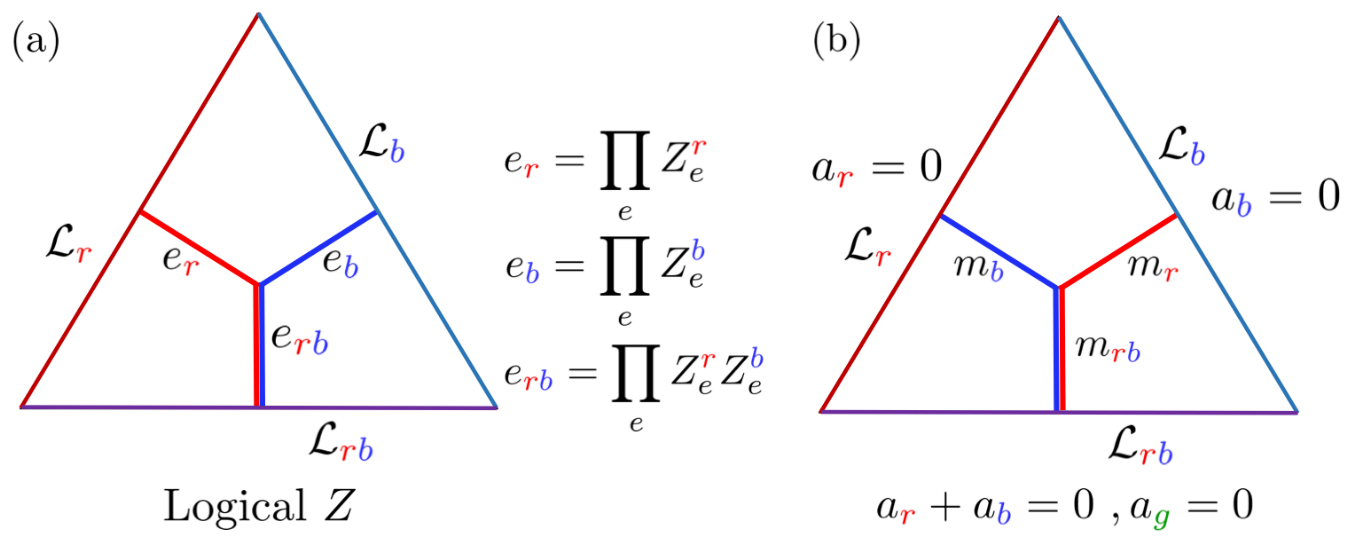
Group outer automorphism — Davydova, Bauer, de la Fuente, Webster, Williamson, Brown (2025)



All above examples can realize logical T gate with proper boundaries

Connection between anyon permutation and logical gates

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Anyon permutation → Design boundaries

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- Different permutations can realize the same logical gate under different boundaries.
- Finding boundaries that realize the desired gates transversally is a crucial question.

work in preparation

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- **Floquet code**: permute anyons via a sequence of local measurements instead of unitaries.

Li, Mitra, ZS (in preparation)

- **Universal quantum computation**: potentially can be realized by anyon permutation + code switching.

Thank You!

Our paper: arXiv: 2602.10110

Kramers-Wannier duality on 1d spin chain

- The Kramers-Wannier (KW) duality:

$$\mathrm{KW}: \ |s_1, s_2, \ldots s_N\rangle \mapsto \sum_{\{t_i\}} \prod_j (-1)^{t_j(s_j+s_{j-1})} |t_1, t_2, \ldots t_N\rangle$$

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- $\mathbb{Z}_2$ symmetry is "gauged": $\eta \mapsto 1$.
- Emergent symmetry: $\hat{\eta} = \prod_i X_i$
- Symmetric local operator maps:

$$X_i \mapsto Z_i Z_{i+1}, \quad Z_i Z_{i+1} \mapsto X_{i+1} \, .$$

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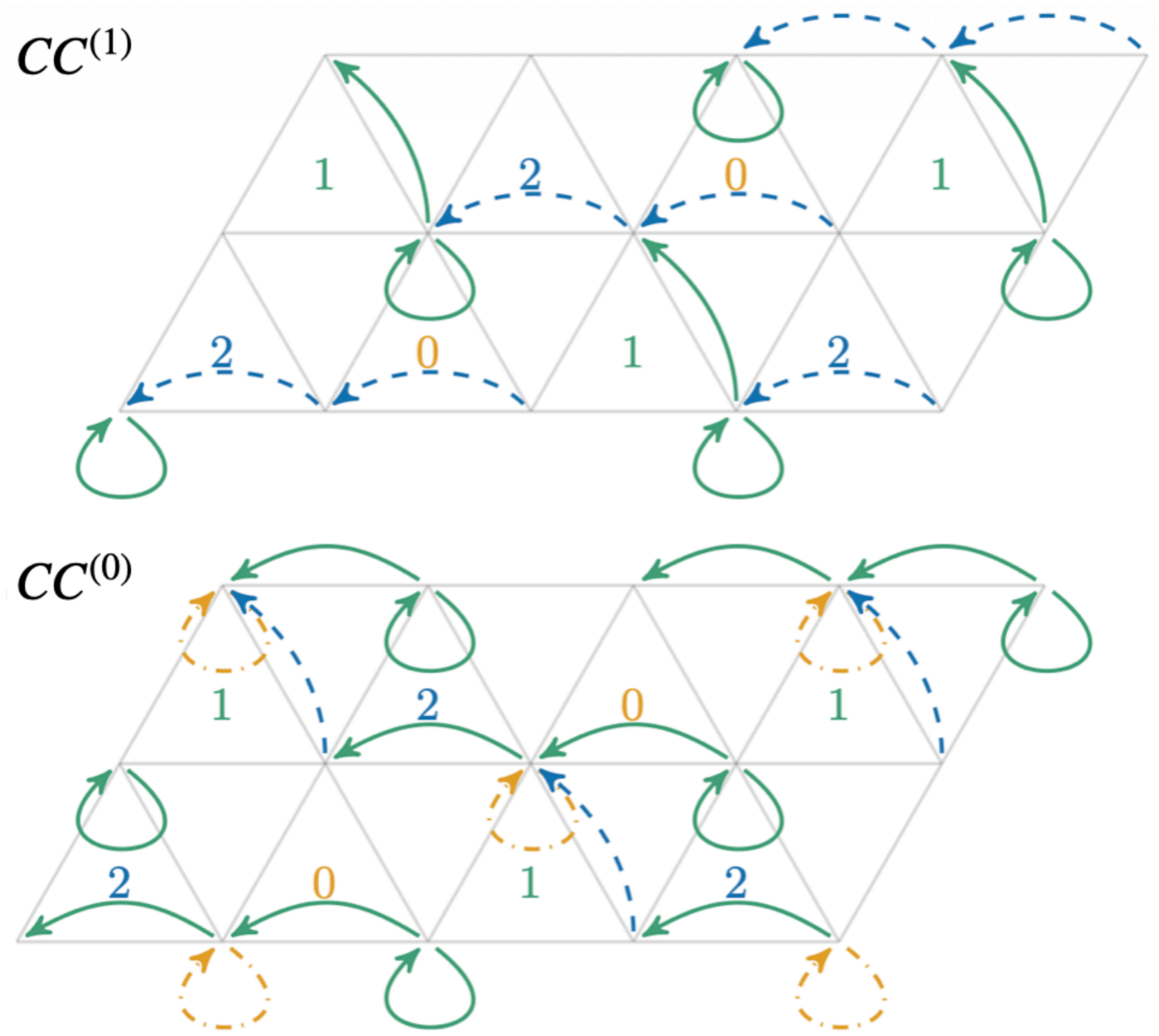
- The KW map is a quantum cellular automata (QCA) on $\mathscr{A}$.
- This map can be realized by a linear depth sequential circuit:

$$S_1 H_1 CZ_{1,2} H_2 CZ_{2,3} \cdots CZ_{N-1,N} H_N S_N$$

Controlled-conjugation circuit



G gauged SPT stacking circuit

$$M_\alpha = \sum_{\{g_i\}} \frac{\alpha(g_*, g_2)\alpha(g_*g_2\bar{g}_3, g_4)\alpha(g_*g_1\bar{g}_6, g_6)}{\alpha(g_*, g_1)\alpha(g_*g_2\bar{g}_3, g_3)\alpha(g_*g_1\bar{g}_6, g_5)} \,|\, \{g_i\} \rangle\langle \{g_i\} \,|\, .$$

The ancillary element g_* can be taken as identity, since this gate does not depend on g_* despite its explicit appearance in the above definition.

Anyon dictionary between $D(D_4)$ and $D^{\alpha}(\mathbb{Z}_2^3)$

$\mathcal{D}(D_4)$	Centralizers	Characters of irreps of centralizers	$\mathcal{D}^{\alpha}(\mathbb{Z}_2^3)$
$([e], 1)$	$\langle r, s\rangle$	trivial	1
$([e], \mathbf{1}_s)$	$\langle r, s\rangle$	$\chi([s]) = \chi([rs]) = -1,\ \chi([e]) = \chi([r]) = \chi([r^2]) = 1$	e_{rg}
$([e], \mathbf{1}_r)$	$\langle r, s\rangle$	$\chi([r]) = \chi([rs]) = -1,\ \chi([e]) = \chi([r^2]) = \chi([s]) = 1$	e_r
$([e], \mathbf{1}_r\mathbf{1}_s)$	$\langle r, s\rangle$	$\chi([s]) = \chi([r]) = -1,\ \chi([e]) = \chi([r^2]) = \chi([rs]) = 1$	e_g
$([e], \pi)$	$\langle r, s\rangle$	$\chi([e]) = 2,\ \chi([r^2]) = -2,\ \chi([r]) = \chi([s]) = \chi([rs]) = 0$	m_b
$([r^2], 1)$	$\langle r, s\rangle$	trivial	e_{rgb}
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$([r], 1)$	$\langle r\rangle$	trivial	m_{rg}
$([r], \omega)$	$\langle r\rangle$	$\chi(r^j) = e^{\frac{j\pi i}{2}}$	s_{rgb}
$([r], \omega^2)$	$\langle r\rangle$	$\chi(r^j) = e^{j\pi i}$	f_{rg}
$([r], \omega^3)$	$\langle r\rangle$	$\chi(r^j) = e^{-\frac{j\pi i}{2}}$	$\bar{s}_{rgb}$
$([s], (1,1))$	$\langle r^2, s\rangle$	trivial	m_{gb}
$([s], (-1,1))$	$\langle r^2, s\rangle$	$\chi(r^2) = \chi(r^2 s) = -1,\ \chi(e) = \chi(s) = 1$	m_g
$([s], (1,-1))$	$\langle r^2, s\rangle$	$\chi(s) = \chi(r^2 s) = -1,\ \chi(e) = \chi(r^2) = 1$	f_g
$([s], (-1,-1))$	$\langle r^2, s\rangle$	$\chi(r^2) = \chi(s) = -1,\ \chi(e) = \chi(r^2 s) = 1$	f_{gb}
$([rs], (1,1))$	$\langle r^2, s\rangle$	trivial	m_{rb}
$([rs], (-1,1))$	$\langle r^2, s\rangle$	$\chi(r^2) = \chi(r^2 s) = -1,\ \chi(e) = \chi(s) = 1$	m_r
$([rs], (1,-1))$	$\langle r^2, s\rangle$	$\chi(s) = \chi(r^2 s) = -1\,\chi(e) = \chi(r^2) = 1$	f_r
$([rs], (-1,-1))$	$\langle r^2, s\rangle$	$\chi(r^2) = \chi(s) = -1,\ \chi(e) = \chi(r^2 s) = 1$	f_{rb}