Lecture 24 of Adrian Ocneanu

Notes by the Harvard group

Lecture notes for 25 October 2017.

Bipartite Graphs For a bipartite graph G, we denote by $V(G)$ the set of vertices of G, and let

$$V(G) = V^0(G) \sqcup V^1(G)$$

be the partition of $V(G)$ into even and odd vertices respectively. We denote by $E(G)$ the set of edges of G. If G is connected and finite, let $\mu_G : V(G) \to \mathbb{R}_+$ denote a Perron-Frobenius eigenvector of G (existence and uniqueness up to positive multiples follows from the Perron-Frobenius Theorem on irreducible matrices).

Plaquettes Let G_1 and G_2 be connected finite bipartite graphs. Let G_3 and G_4 be (not necessarily connected) bipartite multi-graphs with

$$V^0(G_3) = V^0(G_1), \quad V^1(G_3) = V^0(G_2), \quad V^0(G_4) = V^1(G_1), \quad V^1(G_4) = V^1(G_2).$$

A **plaquette** p is a 4-cycle in $G = \bigcup G_j$ which contains an edge from each G_j. Given a choice of $G = \bigcup G_j$, let $\mathcal{P}$ denote the set of all plaquettes. For a plaquette $p \in \mathcal{P}$, we denote the edge of p in G_j by $G_j(p)$. The vertices of p will usually be denoted by v, v', w' and w. Since G_3 and G_4 can be multi-graphs, a plaquette may not be determined by its vertices. However, a plaquette $p \in \mathcal{P}$ is determined by the pair

$$\overline{p} = \big(G_3(p), G_4(p)\big) \in E(G_3) \times E(G_4).$$

Let $\overline{\mathcal{P}} \subset E(G_3) \times E(G_4)$ denote the set of all such pairs. For two vertices $v \in G_3$ and $w \in G_4$, let

$$\mathcal{P}_{vw} = \{p \in \mathcal{P} : v, w \in p\} \subset \mathcal{P}.$$

If $e = \{v, w\}$ is an edge (of either G_1 or G_2), we also denote $\mathcal{P}_{vw}$ by $\mathcal{P}_e$. Since v and w are fixed and G_1 and G_2 are simple graphs, $\mathcal{P}_{vw}$ can be naturally identified with a subset of $E(G_3) \times E(G_4)$. In this way, maps on $\mathcal{P}_{vw}$ may be viewed as matrices (perhaps after filing the missing entries with 0 if $\{v, w\}$ is an edge). Finally, let W be any function

$$W : \mathcal{P} \to \mathbb{C}.$$

We may extend W to a matrix $E(G_3) \times E(G_4) \to \mathbb{C}$ whose support is $\overline{\mathcal{P}}$. If we view G_3 and G_4 as multivalued functions $V(G_1) \to V(G_2)$, inducing a multivalued function $E(G_1) \to E(G_2)$, then W gives $E(G_1) \to E(G_2)$ the structure of a linear map of coordinate spaces $\mathbb{C}^{E(G_1)} \to \mathbb{C}^{E(G_2)}$.

Definition 0.1. *Let G_1 and G_2 be connected finite bipartite graphs. A **q-map** $q : G_1 \to G_2$ is a triple $q = (G_3, G_4, W)$, where G_3, G_4 and W are as above, such that for any two vertices $v \in G_1$ and $w \in G_2$ of opposite parity, the restriction*

$$W : \mathcal{P}_{vw} \to \mathbb{C}$$

is a unitary matrix scaled by $\sqrt{\mu_{G_1}(v)\mu_{G_2}(w)}$.

Q-Map Diagrams Let $q = (G_3, G_4, W) : G_1 \to G_2$ be a q-map, and again let $G = \bigcup_j G_j$. Let $G_1 \times_{\mathbb{Z}_2} G_2$ be the bipartite graph product of G_1 with G_2. For a vertex $\omega = (v, w) \in G_1 \times_{\mathbb{Z}_2} G_2$, let $\bar{\omega}$ denote the set of edges connecting v and w in G. Thus, if v and w are even, then $\bar{\omega} \subset E(G_3)$, and if v and w are odd, then $\bar{\omega} \subset E(G_4)$. We define the **fusion** of q to be the function

$$\mu_q : V(G_1 \times_{\mathbb{Z}_2} G_2) \to \mathbb{N}, \quad \omega \mapsto \mathrm{Card}(\bar{\omega}).$$

We call the full subgraph of $G_1 \times_{\mathbb{Z}_2} G_2$ whose vertices are the support of μ_q the **diagram** $D = D_q$ of q. The edges of D are pairs $\varepsilon \in E(G_1) \times E(G_2)$ obtained as follows; choose two connected vertices of the same parity in G, then move in both G_1 and G_2 to adjacent vertices such that the two new vertices are also connected. The pair consisting of the edges one moved along is an edge of D. If ε connects the vertices ω and ω', we also identify ε with $\{\omega, \omega'\}$.

Rectangular Matrices Let $\varepsilon = \{\omega, \omega'\}$ be an edge of D with $\omega = (v, w)$ and $\omega' = (v'w')$. Then we make the natural identification

$$\bar{\omega} \times \bar{\omega}' = \mathcal{P}_{vw'} \cap \mathcal{P}_{v'w}.$$

Notice that this is also compatible with W. Therefore we may associate to the edge ε the matrix which is the restriction

$$W : \bar{\omega} \times \bar{\omega}' \to \mathbb{C}.$$

Thus, the values which W takes on $\bar{\omega} \times \bar{\omega}'$ can be found in the unitaries on both $\mathcal{P}_{vw'}$ and $\mathcal{P}_{v'w}$.

Forming the Unitaries Notice that the set of all possible unitaries is in bijection with the set O of pairs of vertices of opposite parity

$$O = V(G_1 \times G_2) - V(G_1 \times_{\mathbb{Z}_2} G_2).$$

Let

$$\rho_1 : E(G_1 \times_{\mathbb{Z}_2} G_2) \to O$$

be the projection which takes an edge $\varepsilon = (e, e')$ to the even vertex of e paired with the odd vertex of e'. Let

$$\rho_2 : E(G_1 \times_{\mathbb{Z}_2} G_2) \to O$$

be the projection which takes an edge $\varepsilon = (e, e')$ to the odd vertex of e paired with the even vertex of e'. Fix two vertices $v \in G_1$ and $w' \in G_2$ of opposite parity, and require v to be even. Take the union of all the $\bar{\omega} \times \bar{\omega}'$ such that $\rho_1(\varepsilon) = (v, w')$, where $\varepsilon = \{\omega, \omega'\}$. Then this union is $\mathcal{P}_{vw'}$. Fix two vertices $v \in G_1$ and $w' \in G_2$ of opposite parity, but now require v to be odd. Take the union of all the $\bar{\omega} \times \bar{\omega}'$ such that $\rho_2(\varepsilon) = (v, w')$, where $\varepsilon = \{\omega, \omega'\}$. Then this union is $\mathcal{P}_{vw'}$.

Theorem 0.2. *Let $G_2 = X_n$ for $X \in \{A, D, E\}$, and let $G_1 = A$ be the unique graph of type A with the same norm as G_2 (equivalently the same Coxeter number). Then for any vertex $v \in G$, there exists a unique q-map up to gauge*

$$q_v : A \to G_2$$

such that $(1, v)$ is an edge of G. In particular, there are exactly n many distinct q-maps up to gauge from A into G_2.

Map of Edges and Paths Let $\xi = \{v, v'\}$ be an edge of G_1. Let $(j,k) \in E(G_3) \times E(G_4)$ such that $v \in j$ and $v' \in k$. If it exists, let p_{jk} denote the plaquette such that $\overline{p} = (j,k)$, otherwise let $p_{jk} = 0$. Then we have the matrix

$$\left[\Phi(\xi)\right]_{j,k} = W(p_{jk})G_2(p_{jk}) \in \mathbb{C}^{E(G_2)}$$

where we make the convention that $G_2(0) = 0$. More generally, let $\gamma = (e_1, e_2, \dots)$ be a sequence of edges which forms a path in G_1. Then for each tuple

$$\gamma_{\mathcal{P}} = (p_1, p_2, \dots) \in \prod_j \mathcal{P}_{e_j}$$

such that $p_j \cap p_{j+1}$ contains an edge of $G_3 \cup G_4$, we have the corresponding path in G_2

$$G_2(\gamma_{\mathcal{P}}) = (G_2(p_1), G_2(p_2), \dots).$$

Let $\gamma' = (e'_1, e'_2, \dots)$ be a path in G_2 which is an image of γ. Let $j, k \in E(G_3) \cup E(G_4)$. Denote by $[\gamma, \gamma']_{jk}$ the set of tuples $\gamma_{\mathcal{P}}$ such that $G_2(\gamma_{\mathcal{P}}) = \gamma'$ and $j \in p_1$ and $k \in p_n$. Thus, $[\gamma, \gamma']_{jk}$ is the set of connections between paths which have a fixed boundary. Then we define

$$W\left([\gamma, \gamma']_{jk}\right) = \sum_{\gamma_{\mathcal{P}} \in [\gamma, \gamma']_{jk}} \prod_j \overline{W}(p_j)$$

where $\overline{W}$ denotes the chessboard conjugation of W. Then we have the matrix

$$\Phi(\gamma) = \Big[\sum_{\gamma'} W\left([\gamma, \gamma']_{jk}\right)\gamma' \Big]_{jk}.$$

Remark 0.3. *Let $\gamma = (e_1, e_2, \dots)$ be a path in G_1. Then $\Phi(\gamma)$ is the product of matrices*

$$\Phi(\gamma) = \Phi(e_1)\Phi(e_2)\dots.$$

More generally, let $\gamma = \gamma'\gamma''$, then

$$\Phi(\gamma) = \Phi(\gamma')\Phi(\gamma'').$$

Definition 0.4. *Let $X \in \{A, D, E\}$, and let A be the graph of type A with the same norm as X_n. An **essential path** (up to gauge) in X_n issuing from $v \in V(X_n)$ is any linear combination of entries of the matrix $\Phi_{q_v}(\gamma)$, where γ is a path in A without backtracks which starts at 1.*

The choice of unitaries in a q-map naturally determines a basis for each space of essential maps. The n-many q-maps (up to gauge) $A \to X_n$ correspond to the n-many vertices an essential path can issue from.

The Ribbon (over sl_2) Let $X \in \{A, D, E\}$. We define the **ribbon** $\mathcal{R}(X_n)$ of type X_n to be the bipartite graph product of the cycle of length $2N$ with X_n, where N is the Coxeter number of X_n. If we pick a placement of the single spherical A_1 mirror, and pick a Weyl chamber, this product can be taken with $\mathbb{Z}/2N$ instead of a cycle. The 0 is the mirror, and the direction $0 \to 1$ is the chamber. Call this the **pointed/orientated ribbon**. A **biharmonic function** on $\mathcal{R}(X_n)$ is a map

$$V(\mathcal{R}(X_n)) \to \mathbb{Z}$$

such that the vertical and horizon sums of adjacent vertices are equal. A biharmonic function is determined by its values on two adjacent levels. The map μ_q of a q-map diagram can be extended uniquely to the ribbon. The μ_q, as q varies over the n-many q-maps, forms a basis for biharmonic maps on the ribbon. We obtain:

Theorem 0.5. *The dimension of the space of biharmonic maps on the ribbon of X_n is n.*

Definition 0.6. *A* ***quiver embedding*** *is an embedding of X in its ribbon such that there is one point in every orbit.*

Quiver Representations We define the **dominant ribbon** to be the bipartite graph product $A_{m+1} \times_{\mathbb{Z}_2} X_n$, where A_m has the same norm as X_n. We define the **strongly dominant ribbon** to be the bipartite graph product $A_m \times_{\mathbb{Z}_2} X_n$, where A_m has the same norm as X_n. If we pick an orientation of $A_{m+1} \times_{\mathbb{Z}_2} X_n$ such that all the arrows point up or down, every Dynkin quiver Q of type X_n can be embedded $Q \hookrightarrow A_{m+1} \times_{\mathbb{Z}_2} X_n$. For every choice of vertex $v \in A_{m+1} \times_{\mathbb{Z}_2} X_n$, we obtain an indecomposable representation of Q on the vector spaces of essential paths which issue from v to Q. We obtain the linear maps by annihilating the final edges of essential paths.

Roots We define a **root** to be a vertex of $\mathcal{R}(X_n)$. We obtain a bijection with the roots of the root system of type X_n as follows. To determine the simple roots, pick an embedding

$$V(X_n) \hookrightarrow V(\mathcal{R}(X_n))$$

such that the distance between even and odd vertices is $N-1$. Then associate to the vertex $v \in \mathcal{R}(X_n)$ the sum of simple roots determined by the dimension of the space of essential paths issuing from v to each simple root (i.e. to each vertex of the image of $V(X_n)$ in the ribbon). Simple roots only have essential paths to themselves since the longest essential path has length $N-2$. The set of choices is in bijection with the set of Coxeter elements in the Weyl group of type X_n.

We can also obtain a bijection with roots by picking an embedding of X_n in the ribbon. Then associate to the vertex $v \in \mathcal{R}(X_n)$ the sum of simple roots determined by the dimension of the space of essential paths issuing from v to each vertex of the image of $V(X_n)$ in the ribbon.

Fusion We can orientate the ribbon (which is automatic if we take the product with $\mathbb{Z}/2N$). Then the fusion up from a vertex $v = (i, \alpha)$, denoted $\mu \uparrow v$, is the biharmonic function with $\mu \uparrow v(v) = 1$ and $\mu \uparrow v(w) = 0$ for $w \in \{(i, \beta), (i-1, \beta)\}$, where $\beta \in X$. The fusion down from v, denoted $\mu_v = \mu \downarrow v$, is the biharmonic function with $\mu \uparrow v(v) = 1$ and $\mu \uparrow v(w) = 0$ for $w \in \{(i, \beta), (i+1, \beta)\}$, where $\beta \in X$. Every vertex on the ribbon has an opposite vertex at distance N. Then

$$\mu_v = \mu \downarrow v = \mu \uparrow v^{op} = -\mu \uparrow v.$$

Inner Product of Roots Let $\mathcal{R} = \mathcal{R}(X_n)$ be the ribbon of type X_n. The inner product of roots is defined as

$$\langle v, w \rangle = \mu \downarrow v(w) - \mu \downarrow v(h^{-1}w).$$

where h is the Coxeter element of $\mathcal{R}$.

Explicit Construction of the Simple Modules over sl_n We give a procedure for extracting the highest weight simple in a tensor product of fundamental representations of sl_n. In this way we obtain explicit constructions of all the simples, in the manner of Gelfand-Tsetlin. The extracted simple has a basis of $\mathbb{N}$-valued functions on the edges of a subdivided equilateral triangle.

Let $V = \mathbb{C}^n$ be the standard representation of sl_n. The k^{th} fundamental representation is $V^{\wedge k}$, equipped with the natural action. Since V is a coordinate space, we have natural bases of the wedge products of V, which can be encoded as 'veggies', formally **upward paths**, growing from the base of a subdivided equilateral triangle. There are $n-1$-many sprouting points, corresponding to the $n-1$-many fundamental representations. We forgetfully associate to each tuple of basis elements the function on edges which counts the number of upward paths containing each edge. The functions form a basis for simples. The simple root vectors $E_{i,i+1} \in sl_n$ act like flippers on upward paths.

Quantum sl_2 at a Root of Unity Let $N = n+1$, so that N is the Coxeter number of A_n. For $k \in \{0, n-1\}$, we denote by σ_k the corresponding simple in $\mathrm{Rep}(U_q(sl_2))$, where q is the $2\mathrm{N}^{\mathrm{th}}$ root of unity (in particular σ_0 is the trivial representation, and $\sigma = \sigma_1$ is the standard representation). More generally, for $k \in \mathbb{Z}$, we may denote by σ_k the image in $\mathrm{Rep}(U_q(sl_2))$ of the simple in $\mathrm{Rep}(U(sl_2))$ of degree k.

Module categories over the fusion category $\mathrm{Rep}(U_q(sl_2))$ are classified by the simply laced Dynkin diagrams, which behave like Cayley graphs for the action of σ_1 on simples. More generally, essential paths of length k tell you what happens when you act on simples by σ_k. We have the following dictionary

$\sigma \otimes -$	edge
$\sigma^{\otimes k} \otimes -$	path of length k
$\sigma_k \otimes -$	essential path of length k.

In general, σ_k is the simple with highest weight in the decomposition of $\sigma^{\otimes k}$ into simples. Therefore, modulo taking the highest weight simple, σ generates the simples. The result of tensoring a simple with the generator is the sum of neighbours $\sigma_k \otimes \sigma \cong \sigma_{k-1} \oplus \sigma_{k+1}$.

Theorem 0.7. *Let $v = (i, \alpha)$ be a vertex on the ribbon $\mathcal{R}$. The projection of the Kronecker delta at v onto the span of the fusion at every vertex of $\mathcal{R}$ is, up to scaling by $\frac{1}{N}$, the fusion at v subtract the fusion at $hv = (i+2, \alpha)$, where h is the Coxeter element of $\mathcal{R}$, thus*

$$\mathrm{proj}\ \delta_v = \frac{1}{2N}\big(\mu_v - \mu_{hv}\big).$$

Proof. Consider the real Hilbert space $\mathbb{R}^{V(\mathcal{R})}$. We have

$$\frac{1}{2N}\langle \mu_{(i,\alpha)} - \mu_{(i+2,\alpha)}, \mu_{(j,\beta)}\rangle =$$

$$\frac{1}{2N}\sum_{(k,\gamma)\in V(\mathcal{R})}\Big(\big(\dim \mathrm{Hom}[\sigma_{k-i}\otimes\alpha,\gamma] - \dim \mathrm{Hom}[\sigma_{k-i-2}\otimes\alpha,\gamma]\big)\cdot \dim \mathrm{Hom}[\sigma_{k-j}\otimes\beta,\gamma]\Big) =$$

$$\frac{1}{2N}\sum_{(k,\gamma)\in V(\mathcal{R})}\big(\dim \mathrm{Hom}[\sigma_{k-j}\otimes\beta,\gamma]\cdot \dim \mathrm{Hom}[\gamma,\alpha_{(k-j)}\otimes\alpha] - \dim \mathrm{Hom}[\gamma,\sigma_{k-i-2}\otimes\alpha]\big) =$$

$$\frac{1}{2N}\sum_{k\in\mathbb{Z}/2N}\big(\dim \mathrm{Hom}[\sigma_{k-j}\otimes\beta,\sigma_{k-i}\otimes\alpha] - \dim \mathrm{Hom}[\sigma_{k-j}\otimes\beta,\sigma_{k-i-2}\otimes\alpha]\big) =$$

$$\frac{1}{2N}\sum_{k\in\mathbb{Z}/2N}\big(\dim \mathrm{Hom}[\sigma_{k-i}\otimes\sigma_{k-j}\otimes\beta,\alpha] - \dim \mathrm{Hom}[\sigma_{k-i-2}\otimes\sigma_{k-j}\otimes\beta,\alpha]\big)$$

where the final equality follows by the tonsorial property of Hom. The sums respect the grading. By biharmonicity of fusion, we have

$$\sigma_{N-1+k} = -\sigma_{N-1-k}.$$

In particular, $\sigma_{N-1} = 0$. Then one can show (alternating Laplacian of A_1) that

$$\frac{1}{2N}\sum_{k\in\mathbb{Z}/2N}\big(\dim \mathrm{Hom}[\sigma_{k-i}\otimes\sigma_{k-j}\otimes\beta,\alpha] - \dim \mathrm{Hom}[\sigma_{k-i-2}\otimes\sigma_{k-j}\otimes\beta,\alpha]\big) =$$

$$\frac{1}{2N}\sum_{k\in\mathbb{Z}/2N}\big(\dim \mathrm{Hom}[-\sigma_{k-i-1-((k-j)+1)}\otimes\beta,\alpha] - \dim \mathrm{Hom}[\sigma_{k-i-1-(-(k-j)-1)}\otimes\beta,\alpha]\big) =$$

$$\frac{1}{2N}\sum_{k\in\mathbb{Z}/2N}\big(\dim \mathrm{Hom}[-\sigma_{j-i-2}\otimes\beta,\alpha] - \dim \mathrm{Hom}[\sigma_{2k-i-j}\otimes\beta,\alpha]\big) =$$

$$\frac{1}{2N}\sum_{k\in\mathbb{Z}/2N}\big(\dim \mathrm{Hom}[\sigma_{i-j}\otimes\beta,\alpha] - 0\big) =$$

$$\dim \mathrm{Hom}[\sigma_{i-j}\otimes\beta,\alpha] =$$

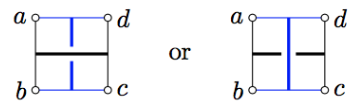
$$\langle \delta_{(i,\alpha)}, \mu_{(j,\beta)}\rangle.$$

□

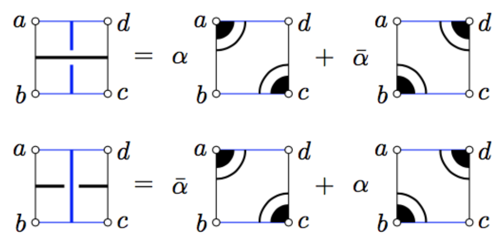
The Davis Complex We define a **cell complex** to be a CW-complex with injective gluing maps (such a CW-complex is sometimes called regular). Cell complexes are rigid with respect to their poset of cells (ordered by inclusion), and the posets which arise as posets of cells of cell complexes are easily characterized (work of A. Bjorner). Thus, cell complexes are equivalent to certain posets. Let $W = (W, S)$ be an affine Coxeter system. Then for $s \in S$, let $J_s = S - \{s\}$. Let $W_{J_s} = \langle J_s\rangle$, which is naturally a finite Coxeter system. The **Davis complex** $\Sigma_D(W)$ of W is the cell complex whose poset of cells is the set of finite standard cosets of W ordered by inclusion. Since W is affine, this is simply the dual of the Coxeter complex.

We can metrize the top dimensional cell of type J_s as a Euclidean polytope by identifying it with the convex hull of the orbit of the barycentre of a simplex in the Coxeter complex of W_{J_s}. Then the induced global length metric on $\Sigma_D(W)$ is isometric to Euclidean space.

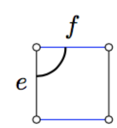
Let G be an $A-D-E$ graph, and take four copies of G in a bi-unitary connection $G-G$. Take a placette in this connection (which will be determined by the vertices), with the braiding,



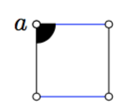
The edges of the placette come from tensoring with the generating simple σ of sl_2 in the quantum module of type G. For example $a-d \in \mathrm{Hom}[\sigma \otimes a, d]$. So the braided wires of the placette corresponding to σ, and the crossing point to an intertwiner. We now normalize the braided placette as follows,



where,

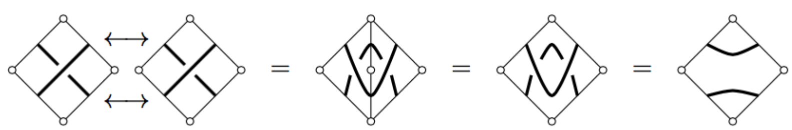


$:= \delta_{e,f}$

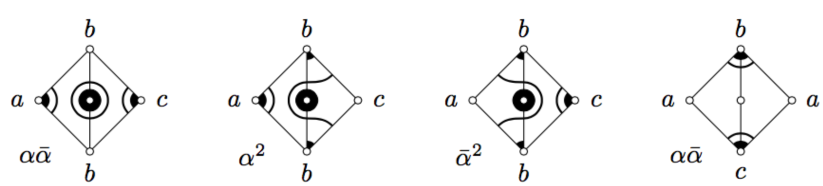


$:= [a]^{\frac{1}{2}}$

For bi-unitarity, we have,



So we obtain the following four terms,



In the middle of the first term we have,

$$\Sigma_d[d]\#\mathrm{edge}(d,b) = (\Delta\mu)(b) = [2]\mu(b)$$

so the total will be,

$$[a]^{\frac{1}{2}}[c]^{\frac{1}{2}}[b][2].$$

Similarly, for the second term we have,

$$[a]^{\frac{1}{2}}[c]^{\frac{1}{2}}[b]\alpha^2.$$

For the third term we have,

$$[a]^{\frac{1}{2}}[c]^{\frac{1}{2}}[b]\bar{\alpha}^2.$$

For the fourth term we have,

$$[b][c]\alpha\bar{\alpha}.$$

So we require,

$$\alpha\bar{\alpha} = 1$$
$$\alpha^2 + \bar{\alpha}^2 + [2] = 0$$

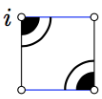
so that,

$$[1] + [2] + [3] = 0.$$

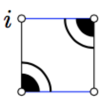
Thus, we obtain the following solutions,

$$\alpha^2 = -e^{\frac{\pm\pi i}{N}}$$
$$\alpha = \pm ie^{\frac{\pi i}{2N}}$$
$$\alpha = -ie^{\frac{-\pi i}{2N}}.$$

Let us compute α in the case of $G = D_4$. We always have,



and,



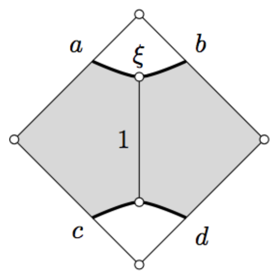
will give us $\sqrt{3}\ \delta_{ij}$. So we have the 3×3 matrix,

$$\begin{bmatrix} \alpha + \sqrt{3}\ \bar{\alpha} & \alpha & \alpha \\ \alpha & \alpha + \sqrt{3}\ \bar{\alpha} & \alpha \\ \alpha & \alpha & \alpha + \sqrt{3}\ \bar{\alpha} \end{bmatrix}$$

We obtain,

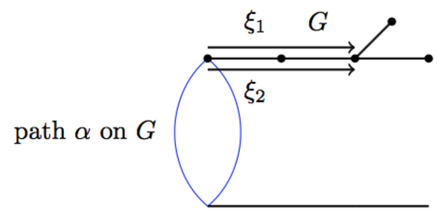
$$\alpha = e^{i\frac{5}{12}\pi}.$$

Again let G be an $A-D-E$ graphs. We build a Hopf algebra as follows. Let $a, b \in G$ and $c, d \in G'$ be vertices, and let 1 denote the fundamental simple of sl_2. Let $\xi \in \mathrm{Hom}[\sigma \otimes b, a]$. Then we let G and G' form a boundary of a region of $SU(2)$ as follows,



In the shaded region we have the usual computations of TQFT over $SU(2)$.

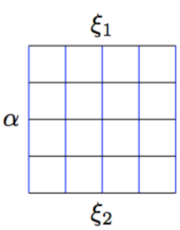
We can view G as a manifold. Take the product of G with a closed path of G,



We then have parallel transport of α along the pair of paths ξ_1 and ξ_2,

$$\mathrm{Transp}_{\xi_1 \xi_2}(\alpha) = \sum_{\beta} \text{coef } \beta$$

where coef β is given by the discrete form of the action,

coef β $=$ α  β $= \sum_{intern} \prod (\square)$.

So we have a notion of flatness. The dimensional of G at a point is the Perron-Frobenius