

Lecture 31 of Adrian Ocneanu

Notes by the Harvard Group

Lecture notes for 10 November 2017.

The two generators of $sl(3)$ are

$$g_1 = u_1 + u_2^{-1} + u_2 u_1^{-1}, \quad g_2 = u_2 + u_1^{-1} + u_1 u_2^{-1},$$

where u_1 and u_2 are showing in Figure 1. They all commute and have common eigenvectors for translation, which is the higher Coxeter elements.

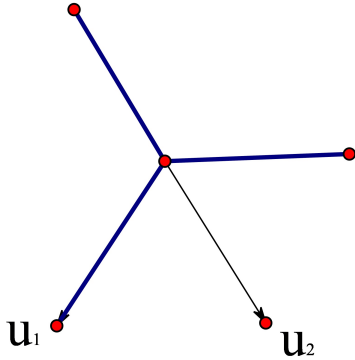


Figure 1: Figure 1

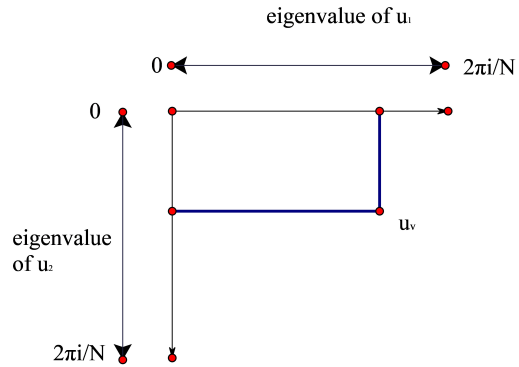


Figure 2: Figure 2

Figure 2 shows the eigenvalues of u_1 and of u_2 . They are supposed to be the interval from 0 to $\frac{2\pi i}{N}$. The eigenvalues are $\exp(\frac{2\pi i}{N} \langle u_k, u_v \rangle)$. Here u_v gives the eigenvalue and is a weight. The inner product belongs to $\frac{1}{3}\mathbb{Z}$. The interval in the picture can be viewed as a kind of the log of the eigenvalue. Note that u_1 and u_2 are not orthogonal. The correct way to plot them is to deform the picture to make it geometrically correct. Then the Weyl group looks like Figure 3.

We may see 12 eigenvalues u_v which has been highlighted, and also 5 more reflections of them. All together there are $12 \times 6 = 72$ joint eigenvectors. In this case, the group is a star (see Figure 4). The list is showing at 06:29 in the video, where the logarithms over $2\pi i$ are shown.

The following contains more explanation on Figure 3. For $sl(3)$ here, we have 8 roots of unity, and the length of the mirror is 8. The red points are between 3 mirrors, which are the cutoffs. They are representations. This red region is the main Weyl chamber. The cutoff becomes an alcove, and these alcoves are repeated by reflection in the mirrors. The eigenvalue of G (which is the star in figure 4). We can simply write them as sums of unitaries. Everything commutes, because g_1 and g_2 are operators of multiplication by the generators. When the generators which we multiply commute,

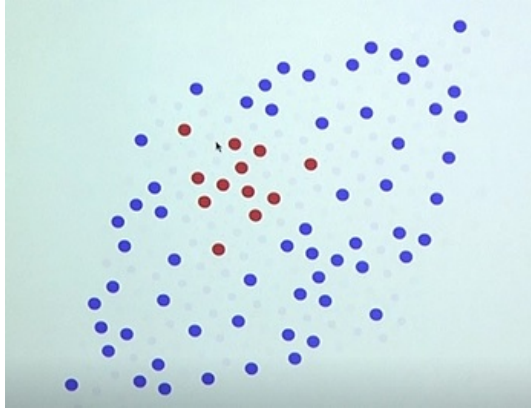


Figure 3: Figure 3

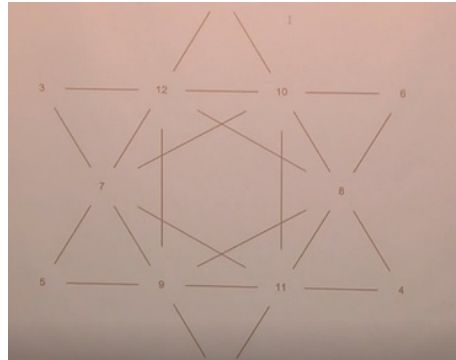


Figure 4: Figure 4

there is a braiding for quantum groups. Whenever you tensor two things, you can braid them to the tensoring in the opposite order, and they will give you the same bi-products. We write these tensoring with generators as a sum of unitaries according to the respective representations. For instance, for u_1 we take the three vectors $u_1, u_2 u_1^{-1}$ and u_2^{-1} (see 10:45 in the video). Now we get some unitaries which commute. If we take the ribbon, which is the product of these with weights, and we put each unitary as multiplying in the direction, then we get the higher Coxeter elements as translations. In the higher case, the invariant subspaces for translations are exactly the exponents. For $sl(2)$ we have 2 translations. Since eigenvalue of unitaries are complex numbers of modulus 1, the proper way to plot them is to take the logarithm divided by $2\pi i$. For example, here is $2\pi i/N$, and $N = 8$ in the case of a star. The star has level 5. There are five things in the red, which you get from the trivial representation up to something of degree 5. In physics this is called level, i.e. the highest degree which is not killed. Then you have the period, which is the period in the mirror, and is additive. Remember that what you see in the Lie group has both a multiplicative and an additive structure. This should be a conclusion of this course: the additive one has mirrors and reflections, so it refers to the diagonal algebra (the diagonal algebra is abelian, so it has some nice additive eigenvalues); the multiplicative part consists of those points that become irreducible representations which are tensored. They have different unit which is the trivial representation (a corner). The three corners are the group $\mathbb{Z}/3\mathbb{Z}$. The eigenvalues come from u_1 and u_2 . They correspond to weights. If we consider some other u_v , we take u_v inner product with u_1 and with u_2 . That gives you two coordinates. These coordinates are not orthogonal. By stretching the picture to make it geometric, you can get exactly the picture. This explains the spectral structure of the ribbon, and also the spectral structure of the graphs. The star we have drawn has this as spectrum. Then you can add the exponentials and get the spectrum as seen if you just take g_1 and g_2 . But the proper way to plot the spectrum is with u_1 and u_2 .

(Answering a question from 15:30 to 17:15.)

Now we move on to the construction of the higher diagonal. For the usual Lie algebra $sl(2)$, we have roots, ribbon and diagonal as in Figure 5. We view it as reflection in some mirrors. We consider G as in Figure 6; it is a cutoff.

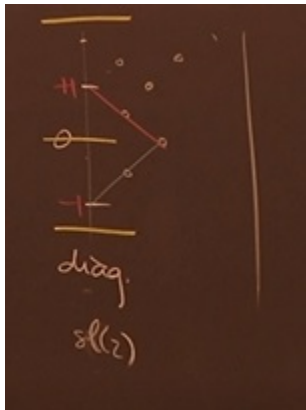


Figure 5: Figure 5

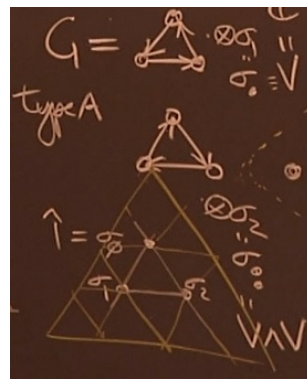


Figure 6: Figure 6

The above example shows the case $L = 1$, $N = 1 + 3 = 4$. We have the fourth root of unity, and $[1] = 1, [2] = \sqrt{2}, [3] = 1, [4] = 0$. The ribbon here is the product we $sl(3)_4 \times \text{Vert}(G)$, where

$\text{Vert}(G)$ is the set of vertices of G . We orient them along the tensor product with σ_1 . How do you write the Cartesian product? One of the way to do it is to copy the graph many times. If you have the Cartesian product with two things, you just put one of them over the other. You could also make a big triangle in which you put the roots and the vertices. The right-hand side graph is the diagonal. The period in Figure 7 is 4. You can see 4 weights, and it is a torus. The two-sided triangles are glued together. When we glue the torus, we also need to apply one of the rotation elements. Remember that the outside elements are in $\mathbb{Z}/3\mathbb{Z}$. In this case we have level 1 and get exactly $\mathbb{Z}/3\mathbb{Z}$. The diagonal would be 4×4 . This is the diagonal of our higher matrices. Now to make the correspondence between them, we use one of the coordinates.



Figure 7: Figure 7

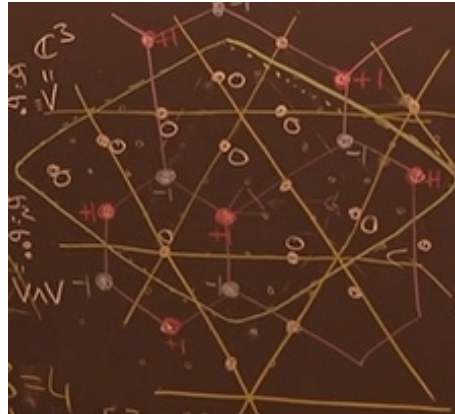


Figure 8: Figure 8

We can put the period of weights in the middle for instance (see Figure 8). Here the roots are the thick white points and the weights not in the root lattice are thin points. In the case of $sl(4)$, the roots and weights looks different. If you put the roots by themselves, you get this picture. They are not at the same scale.

Now we take one of the points (in Figure 8) and write a corresponding diagonal element. As in the usual case, we first build the mirrors. The mirrors are centered at a weight. The graph in the middle is exactly of type A_n .

Now it is important that we have a weight (the red point in the middle) in our graph which is not just a weight but in the root lattice. In case of $sl(2)$ the roots are integers in physics and the weight are half integers.

We write +1 for this red point in the middle. By reflection we get the other ± 1 's. We use green lines to emphasize our period. We have exactly 6 elements in it. The inner product of the root with itself gives 6. This is the vector that realizes a root. The other numbers on the diagonal is 0, as shown in the picture. Hence the diagonal is a 4×4 torus. The roots have $3! = 6$ of ± 1 's; it is the order of the Weyl group. We may draw the hexagons, which are perpendicular to the mirrors. You can recognize here that each of this elements around center of the mirrors form exactly one of the weight permutohedra. These are the ones appear in the theory of Hermann Weyl.

What we want to show is that the inner product between two roots which we have defined by our root formula is exactly the inner product between the corresponding vectors. Let us define the map now. For technical reasons we are going to define the following: for a root on a ribbon, let i be a

weight of \mathfrak{g} and $v \in \text{vert}G = \text{vert}A = \text{weights of } \mathfrak{g} \text{ cut off at Coxeter } N$; we are going to map $i \times v$ to mirrors positioned at the affine Weyl group centered at $-i$ and reflect the weight $\rho + v$ with it, with signs. Here v is the highest weight and ρ is the additive center.

The inner product $\langle (i, v), (j, u) \rangle$ on the ribbon equals the inner product between the higher $\langle h_{i,v}, h_{j,u} \rangle$. Now, with i being the position on the mirrors, $i + \rho + v$ should have parity 0, that is, it should be in the root lattice. The center of the mirrors may be a weight. In the case of $sl(3)$ it is a root, but for $sl(4)$ it is not a root. Moreover, if in the end you add the pebble in the higher kaleidoscope, it should be in the root lattice and reflective. Now we prove this.

$$\begin{aligned}
& \langle (i, v), (j, u) \rangle \\
&= \sum_{w \in W} \epsilon(w) \cdot \text{fusion}(i + w\rho, v)(j, u) \\
&= \sum_{w \in W} \epsilon(w) \# \text{Hom}[\sigma_{j-i-w\rho} \otimes \sigma_v, \sigma_u] \\
&= \sum_{w \in W} \epsilon(w) \sum_{x \in \text{weights of } \sigma_v} \# \text{Hom}[\sigma_{j-i-w\rho} + x, \sigma_u] \\
&= \sum_{w \in W} \epsilon(w) \# \text{Hom}[\sigma_{j-i-w(\rho+v)}, \sigma_{-j+u}].
\end{aligned}$$

As we use Kac-Walton formula, each of the weights is going to be brought to the main Weyl alcove with signs \pm . (see 51:28.) We link all the others by reflections and check whether the number of homs are same. We get the ± 1 precisely if $-i + w(\rho + v)$ belongs to the weight permutohedron of $\rho - j + u$. This ends the proof.

A little more explanation here. When do you get such a coincidence? When do you get such a hom? When the left member is in the Weyl alcove, all is good and you get the exact sign. If it is in one of the reflections, the sign is negative. Note that the left-hand side is Weyl and the right-hand side is Kac-Walton. The first one you get by summing the representation and translating it and it is a Weyl numerator. For the other one, when you count the incidence, you use exactly the coincidence with the Weyl permutahedron of the target which is the Kac-Wilton formula, or WZW in physics. This shows that the inner product of these diagonal elements is exactly the inner product in the ribbon.

In the next lesson we will see how to do the same for graphs of typ B, C , and D .