Equiangular Lines and Eigenvalue Multiplicity

Yufei Zhao

MIT

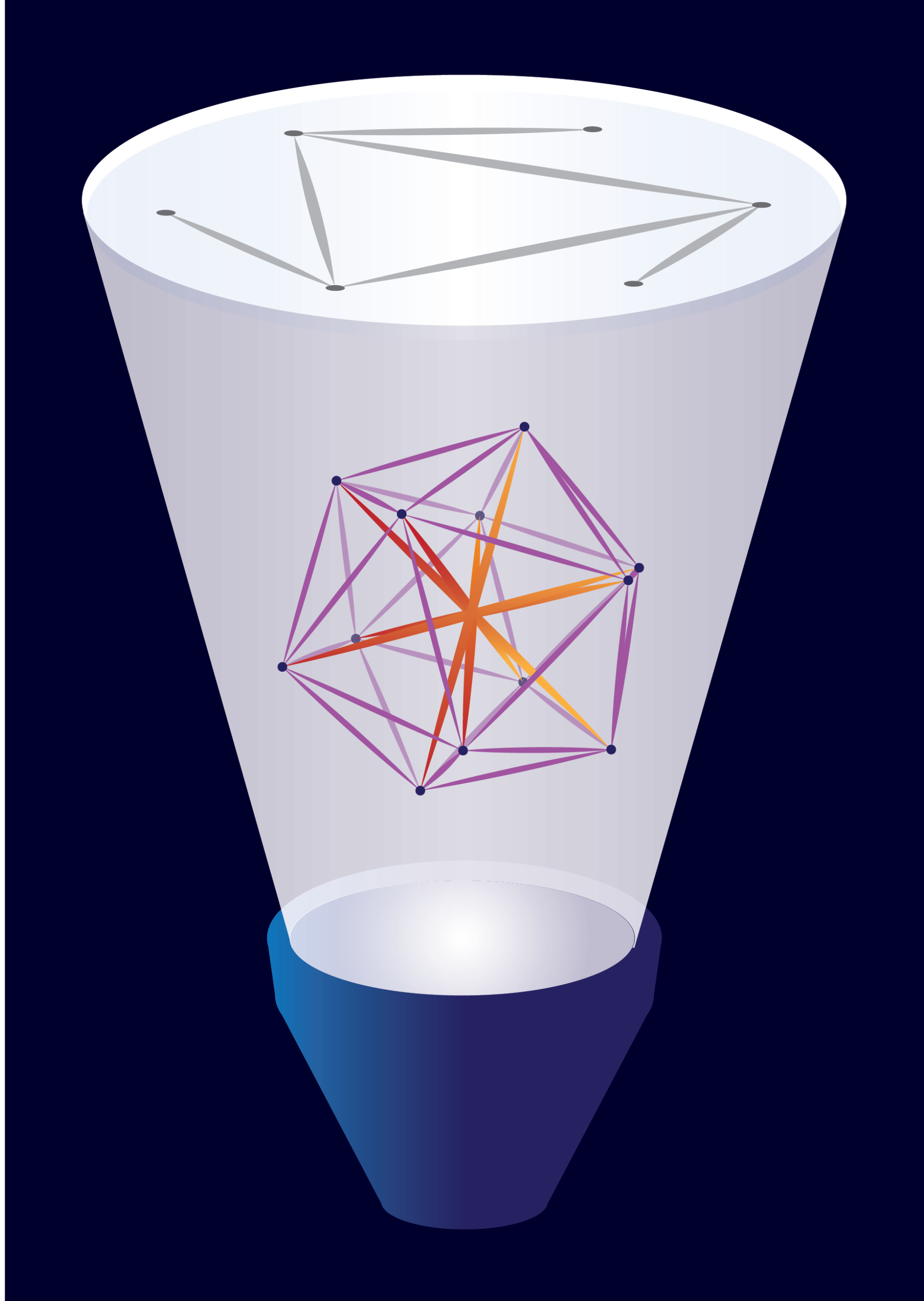
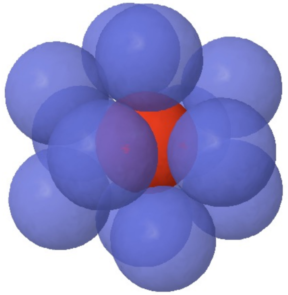
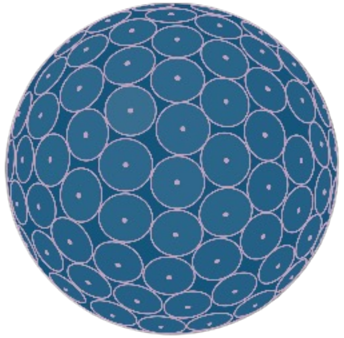
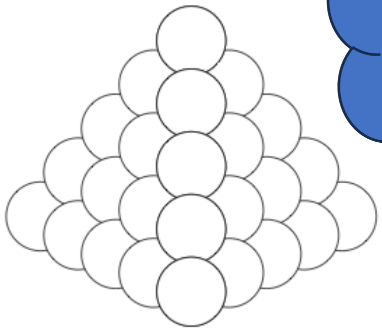


Image credit: Zilin Jiang

Optimal Geometric Arrangement Problems

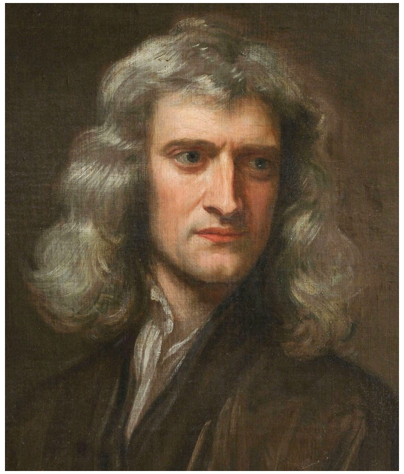






?????? Mystery in high dimensions ??????

Johannes Kepler



Isaac Newton

Carl Friedrich Gauss

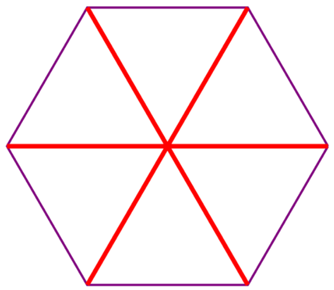
Thomas Hales

Maryna Viazovska

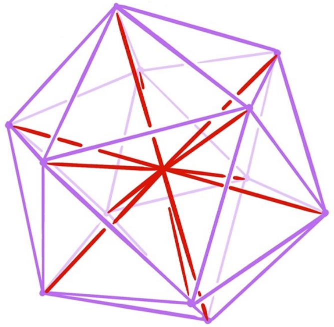
Equiangular Lines

Q: How many equiangular lines can you have in *d* dimensions?

N(*d*) = max # of lines in $\mathbb{R}^d$ with pairwise equal angles



N(2) = 3



N(3) = 6

Exact answer known for finitely many *d*

General bounds:

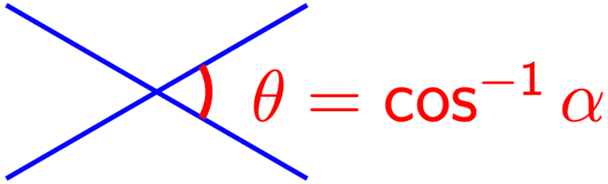
[de Caen '00] $cd^2 \leq N(d) \leq \binom{d+1}{2}$ [Gerzon '73]

In lower bound constructions, pairwise angles $\to 90°$ as $d \to \infty$

Equiangular Lines With a Fixed Angle

Q: Fix an angle, how many equiangular lines can you have with this given angle in *d* dimensions, for large *d*?

$N_\alpha(d)$ = max # of lines in $\mathbb{R}^d$ with pairwise angle



(focus: $\alpha > 0$ fixed, $d \to \infty$)

- $N_\alpha(d)$ grows linearly in *d*
  - in contrast to $N(d) = \Theta(d^2)$

- Problem: determine

$$\lim_{d\to\infty} \frac{N_\alpha(d)}{d}$$

Equiangular Lines With a Fixed Angle: History

[Lemmens, Seidel '73] $N_{1/3}(d) = 2(d-1)$ $\forall d \geq 15$

[Neumaier '89] $N_{1/5}(d) = \left\lfloor \frac{3}{2}(d-1) \right\rfloor$ for sufficiently large d

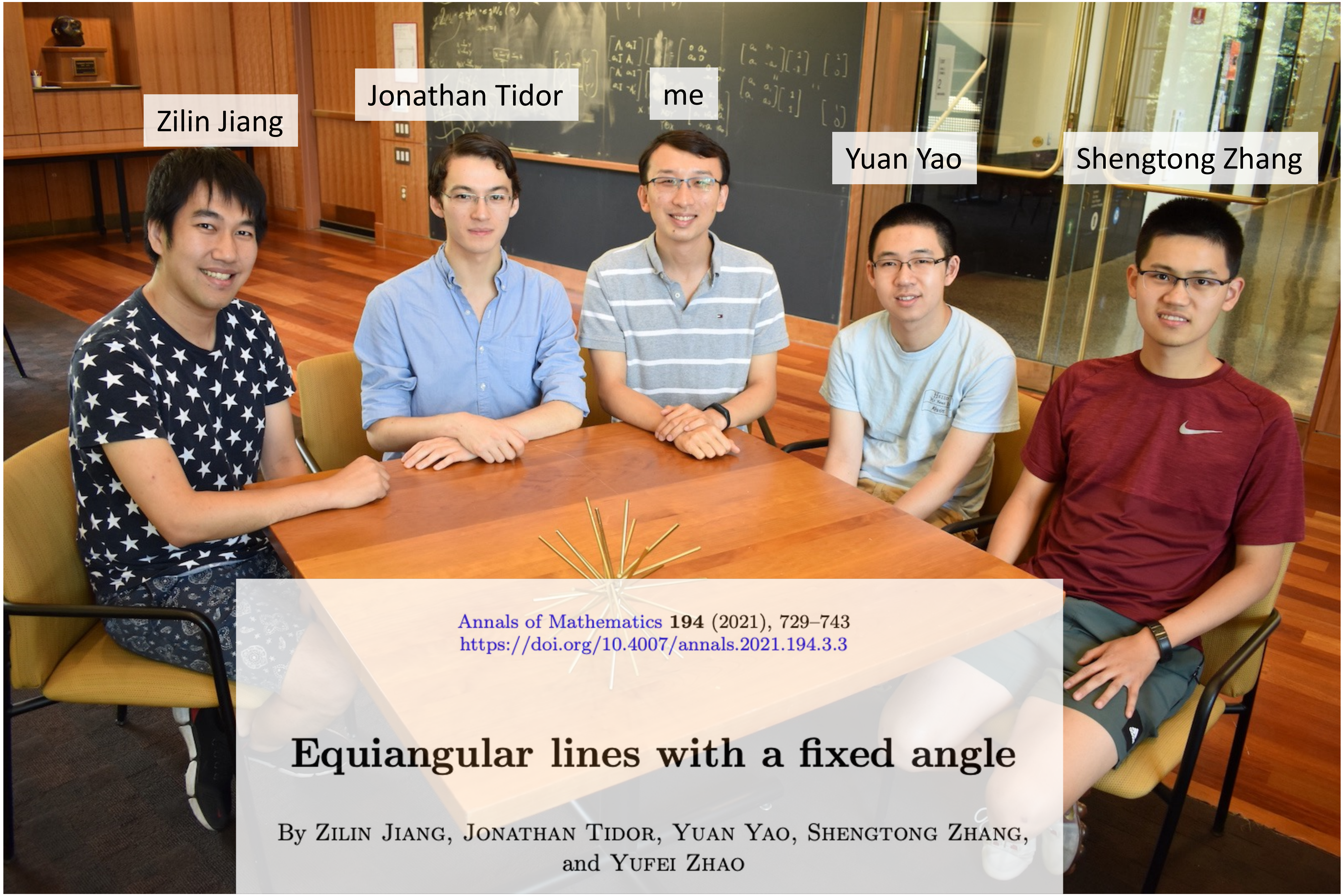
"the next interesting case will require substantially stronger techniques"

[Bukh '16] $N_\alpha(d) \leq C_\alpha d$

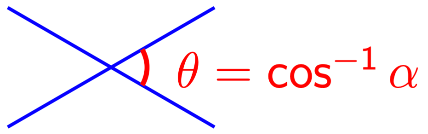
[Balla, Dräxler, Keevash, Sudakov '18] $\limsup_{d\to\infty} N_\alpha(d)/d$ maximized at $\alpha = \frac{1}{3}$

[Jiang, Polyanskii '20] Determined $\lim_{d\to\infty} N_\alpha(d)/d$ $\forall \alpha > 0.196$

[Jiang, Tidor, Yao, Zhang, Z. '21] Solved!


Zilin Jiang
Jonathan Tidor
me
Yuan Yao
Shengtong Zhang
Annals of Mathematics 194 (2021), 729–743
https://doi.org/10.4007/annals.2021.194.3.3
Equiangular lines with a fixed angle
By Zilin Jiang, Jonathan Tidor, Yuan Yao, Shengtong Zhang, and Yufei Zhao

The Answer [Jiang, Tidor, Yao, Zhang, Z. '21]

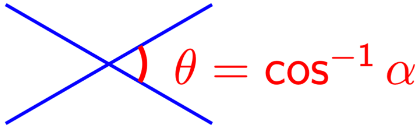


$N_\alpha(d)$ = max # of lines in $\mathbb{R}^d$ with pairwise angle $\cos^{-1} \alpha$

For every integer $k \geq 2$

$$N_{\frac{1}{2k-1}}(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor \quad \forall d \geq d_0(k)$$

The Answer [Jiang, Tidor, Yao, Zhang, Z. ’21]



$N_\alpha(d)$ = max # of lines in $\mathbb{R}^d$ with pairwise angle $\cos^{-1}\alpha$

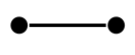
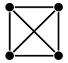
Let $\alpha \in (0,1)$ and $\lambda = \frac{1-\alpha}{2\alpha}$

spectral radius order $k = k(\lambda)$

= min # vertices in a graph with top eigval λ (adjacency matrix)

- If $k < \infty$, $N_\alpha(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor$ $\forall d \geq d_0(\alpha)$
- If $k = \infty$, $N_\alpha(d) = d + o(d)$

Examples

α	λ	k	G	$N_\alpha(d)$ $\forall$ large d
$1/3$	1	2		$2(d-1)$
$1/5$	2	3		$\left\lfloor \frac{3}{2}(d-1) \right\rfloor$
$1/7$	3	4		$\left\lfloor \frac{4}{3}(d-1) \right\rfloor$
$1/(2k-1)$	$k-1$	k	K_k	$\left\lfloor \frac{k}{k-1}(d-1) \right\rfloor$
$1/(1+2\sqrt{2})$	$\sqrt{2}$	3		$\left\lfloor \frac{3}{2}(d-1) \right\rfloor$

Sublinear Second Eigenvalue Multiplicity

Every connected bounded degree graph has sublinear second eigenvalue multiplicity

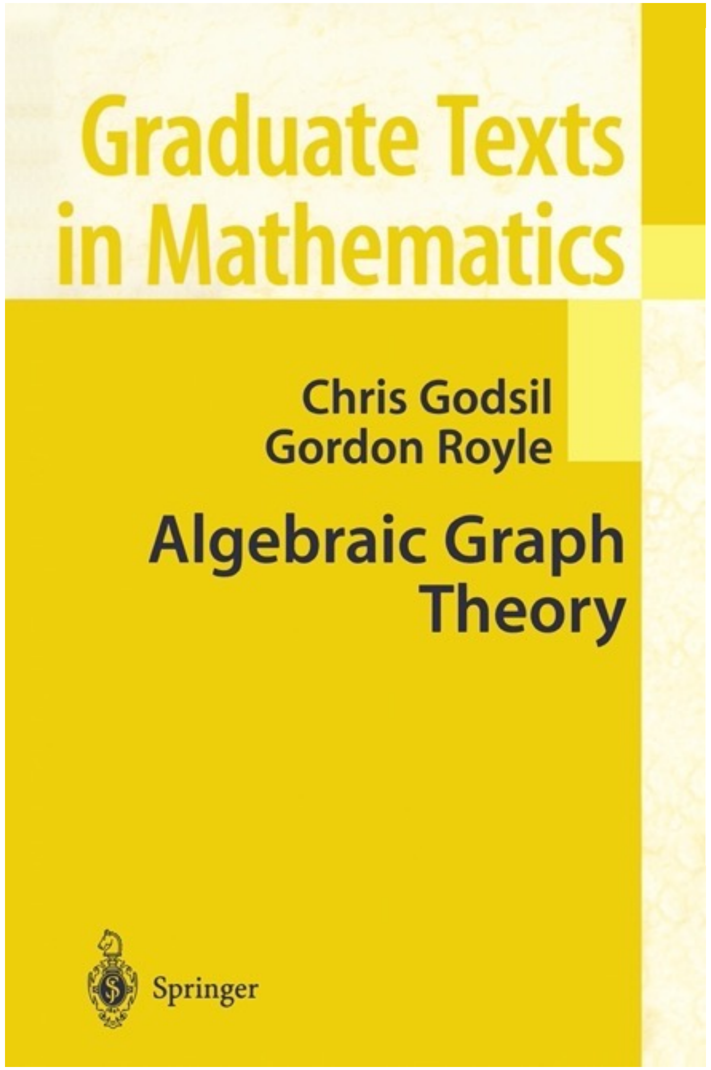
Always referring to the adjacency matrix

Theorem. (Jiang, Tidor, Yao, Zhang, Z. '21)

A connected *n*-vertex graph with maximum degree *Δ* has second largest eigenvalue with multiplicity

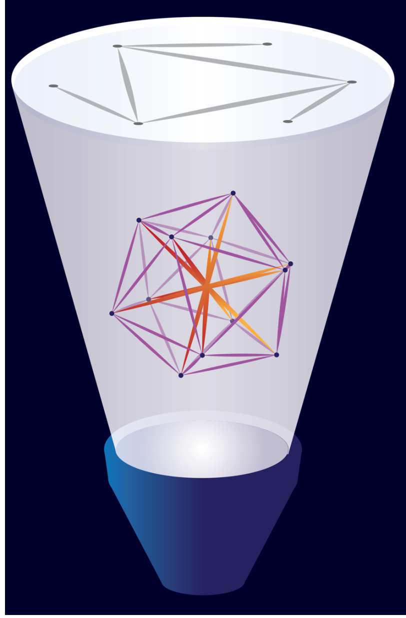
$$\leq C \log \Delta \frac{n}{\log \log n}$$

Connection to Spectral Graph Theory

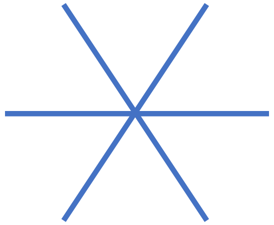


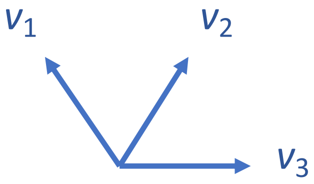
The problem that we are about to discuss is one of the founding problems of algebraic graph theory, despite the fact that at first sight it has little connection to graphs. A *simplex* in a metric space with distance function d is a subset S such that the distance $d(x, y)$ between any two distinct points of S is the same. In $\mathbb{R}^d$, for example, a simplex contains at most $d + 1$ elements. However, if we consider the problem in real projective space then finding the maximum number of points in a simplex is not so easy. The points of this space are the lines through the origin of $\mathbb{R}^d$, and the distance between two lines is determined by the angle between them. Therefore, a simplex is a set of lines in $\mathbb{R}^d$ such that the angle between any two distinct lines is the same. We call this a set of *equiangular lines*. In this chapter we show how the problem of determining the maximum number of equiangular lines in $\mathbb{R}^d$ can be expressed in graph-theoretic terms.

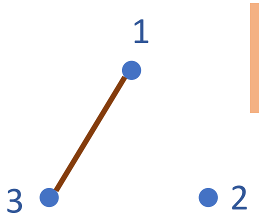
Connection to Spectral Graph Theory



Equiangular lines in $\mathbb{R}^d$ → unit vectors in $\mathbb{R}^d$ → graph G







Edge = obtuse
Non-edge = acute

Given a list of vectors $v_1, \ldots, v_n \in \mathbb{R}^d$, Gram matrix is PSD and rank ≤ d:

$$\text{Gram matrix} = \begin{pmatrix} v_1 \cdot v_1 & \cdots & v_1 \cdot v_n \\ \vdots & \ddots & \vdots \\ v_n \cdot v_1 & \cdots & v_n \cdot v_n \end{pmatrix} = (1-\alpha)I - 2\alpha A_G + \alpha J$$

off-diag entries $\pm\alpha$

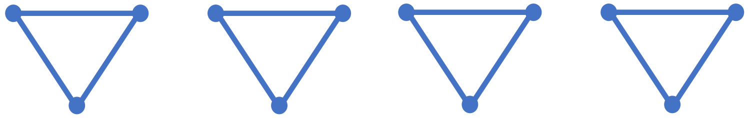
J = all 1s matrix

Equivalent problem: given α, d, find graph G with max # vertices N s.t.

$(1-\alpha)I - 2\alpha A_G + \alpha J$ is PSD and rank ≤ d

Lower Bound Construction

To verify $N_{1/5}(9) \geq 12$, check

$G =$ 

$$(1-\alpha)I - 2\alpha A_G + \alpha J = \begin{pmatrix}
1 & -\alpha & -\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
-\alpha & 1 & -\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
-\alpha & -\alpha & 1 & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha & 1 & -\alpha & -\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha & -\alpha & 1 & -\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha & -\alpha & -\alpha & 1 & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & 1 & -\alpha & -\alpha & \alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & -\alpha & 1 & -\alpha & \alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & -\alpha & -\alpha & 1 & \alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & 1 & -\alpha & -\alpha \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & -\alpha & 1 & -\alpha \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & -\alpha & -\alpha & 1
\end{pmatrix}$$

is PSD and rank 9
$(\alpha = 1/5)$

Upper Bound on N

Problem: Given α, d, find graph G with max # vertices N s.t.
$\text{Gram} = (1-\alpha)I - 2\alpha A_G + \alpha J$ is PSD and rank ≤ d.

By rank–nullity

$$\begin{aligned} N &= \text{rank(Gram)} + \text{nullity(Gram)} \\ &\leq d + \text{null}((1-\alpha)I - 2\alpha A_G + \alpha J) \\ &\leq d + \text{null}\big((1-\alpha)I - 2\alpha A_G\big) + 1 \end{aligned}$$

Multiplicity of $\lambda = \frac{1-\alpha}{2\alpha}$ as an eigval of A_G

Gram is PSD and rank J = 1. If $\lambda = \frac{1-\alpha}{2\alpha}$ is an eigval of A_G, it must be either

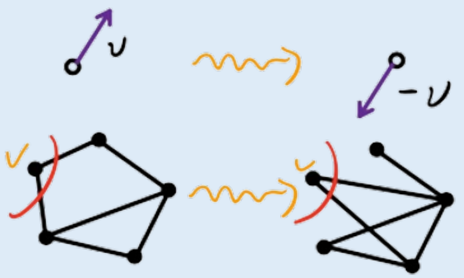
the largest eigval (equality case) *easy*

or

2nd largest (need to rule out for large d) *crux of the problem*

Switching to Bounded Degree

Switching operation:



Theorem (Balla, Dräxler, Keevash, Sudakov '18)

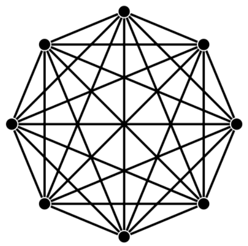
$\forall \alpha\ \exists \Delta = \Delta(\alpha)$: can switch so that max degree $\leq \Delta$

Sublinear Second Eigenvalue Multiplicity

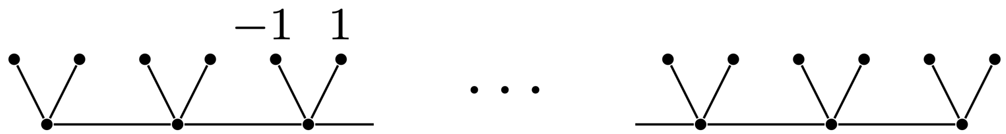
Theorem. (Jiang, Tidor, Yao, Zhang, Z. '21)
Every connected *n*-vertex graph with max deg *Δ* has

$$\mathrm{mult}(\lambda_2, G) = O_\Delta\left(\frac{n}{\log\log n}\right)$$

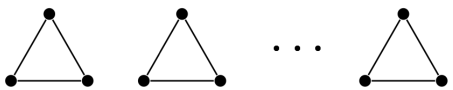
Near miss examples



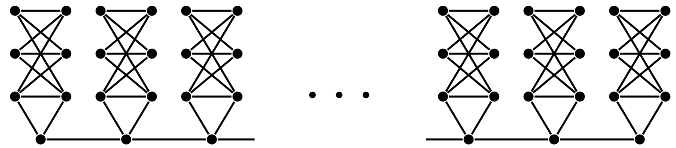
Complete graph
(not bounded degree)



$\mathrm{mult}(0, G) = \Theta(n)$ (not λ_2)



(not connected)



Least eigval: $\mathrm{mult}(-3, G) = \Theta(n)$

Proof ideas of $\mathrm{mult}(\lambda_2, G) = O_\Delta(n/\log\log n)$

Use moments to bound multiplicity:

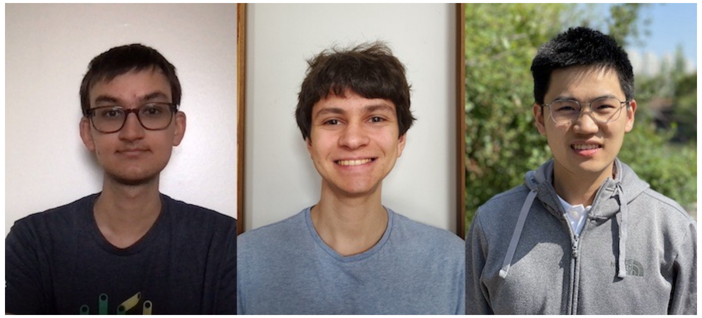
$$\mathrm{mult}(\lambda, G)\lambda^{2s} \le \sum_i \lambda_i(G)^{2s} = \mathrm{tr}\, A_G^{2s}$$

How good is this bound?

Key insight. Delete some vertices from *G*.

1. $\mathrm{mult}(\lambda, G)$ does not change by more than the number of deleted vertices (due to Cauchy eigenvalue interlacing theorem), and
2. RHS decreases significantly if every vertex of *G* is close to one of the removed vertices

Graphs with High Second Eigenvalue Multiplicity



Milan Haiman   Carl Schildkraut   Shengtong Zhang

[Haiman, Schildkraut, Zhang, Z. '22]

- ∃ infinite family of bounded degree graphs with

$$\mathrm{mult}(\lambda_2, G) \geq \sqrt{\frac{n}{\log_2 n}}$$

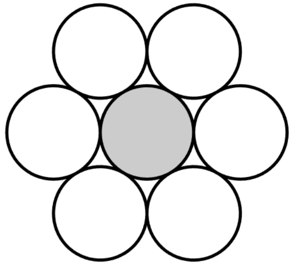
  - Relies on group representations to get multiple eigval. Barrier at $\sqrt{n}$

- Also constructed connected bounded degree graphs

  with $\gtrsim \frac{n}{\log\log n}$ eigenvalues above $\left(1 - \frac{1}{\log n}\right)\lambda_2$

Spherical Codes

- $L \subseteq [-1,1)$. An L-code in $\mathbb{R}^d$ is a set of unit vectors whose pairwise inner products lie in L [Delsarte Goethals Seidel '91]
- $N_L(d)$ = size of largest L-code in $\mathbb{R}^d$
- E.g., points on a sphere with pairwise angle $\geq \theta$: $L = [-1, \cos\theta]$
  - Kissing number $\mathbb{R}^d$: $L = \left[-1, \frac{1}{2}\right]$
  - Linear programming bound (Delsarte '73)
- **Equiangular lines:** $L = \{-\alpha, \alpha\}$



Beyond Equiangular Lines

- Spherical two-distance sets: $L = \{-\beta, \alpha\}$ with $\alpha, \beta > 0$
  [Jiang, Tidor, Yao, Zhang, Z. '21] [Jiang Polyanski '21+]

- Uniacute spherical codes: $L \subset [-1, -\beta] \cup \{\alpha\}$
  [Lepsveridze, Saatashvili, Z., '23+]

Saba Lepsveridze

Aleksandre Saatashvili

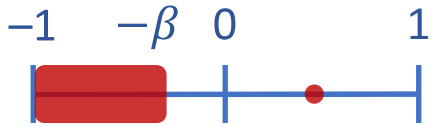
- Framework. Partial results. Lots of open problems
- *Missing piece:* a useful generalization of
  sublinear second eigenvalue multiplicity

Uniacute Spherical Codes

$N_L(d)$ = size of largest L-code in $\mathbb{R}^d$, i.e., unit vec, pairwise inner products lie in L

- [Bukh '15] For $L = [-1, -\beta] \cup \{\alpha\}$ with $\alpha, \beta > 0$

$$N_L(d) = O_L(d)$$



- **Open Problem.** Determine $\lim_{d \to \infty} N_L(d)/d$

- [Balla, Dräxler, Keevash, Sudakov '18] For $L = [-1, -\beta] \cup \{\alpha\}$,

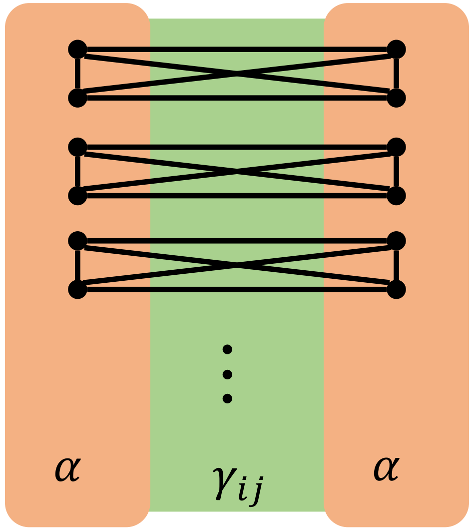
$$\limsup_{d \to \infty} N_L(d)/d \leq 2(1 + \lfloor \alpha/\beta \rfloor)$$

- **Q.** For which (α, β) is this upper bound tight?
  - [Lepsveridze, Saatashvili, Z., '23+] Complete answer

Structure & Modularity – Uniacute Codes [Lepsveridze, Saatashvili, Z.]

Global Structure Theorem. Every L-code with $L \subset [-1, -\beta] \cup \{\alpha\}$ in high dimension has an approximate block structure:

- at most $1 + \lfloor \alpha/\beta \rfloor$ parts (after deleting $O(1)$ vectors)
- *Except on a bounded degree graph:*
- Inner products α within each part
- Inner products $\approx \gamma_{ij} \leq -\beta$ between i^{th} and j^{th} part



Modularity Conjecture.

Optimal uniacute spherical codes are "modular": roughly speaking, by adding modular components to the above template

- True for equiangular lines & all solved cases
- Would follow from an extension of sublinear second eigenvalue multiplicity

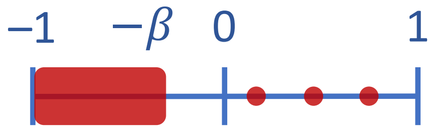
Multiacute Spherical Codes

$N_L(d)$ = size of largest L-code in $\mathbb{R}^d$, i.e., unit vec, pairwise inner products lie in L

[Balla, Dräxler, Keevash, Sudakov '18]

For fixed $L = [-1, -\beta] \cup \{\alpha_1, \ldots, \alpha_k\}$ with $\beta > 0$

$$N_L(d) = O_L\left(d^k\right)$$



Problem: determine $N_L(d)$ more precisely

Complex Equiangular Lines

Zauner's conjecture: $N^{\mathbb{C}}(d) = d^2$ for all *d* (known: $N^{\mathbb{C}}(d) \leq d^2$)

i.e., $\exists\ d^2$ unit vec. in $\mathbb{C}^d$ whose pairwise inner products have equal abs

- "SIC-POVM" from quantum mechanics
- Verified in small dimensions

Restricted angles. Determine $\lim_{d\to\infty} N_\alpha^{\mathbb{C}}(d)/d$

Equiangular *k*-dim subspaces in $\mathbb{R}^d$

Equiangular lines and eigenvalue multiplicities

Equiangular lines with a fixed angle.

$N_\alpha(d)$ = max # of lines in $\mathbb{R}^d$ with pairwise angle $\cos^{-1}\alpha$

∀ integer $k \geq 2$, $N_{\frac{1}{2k-1}}(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor \quad \forall d \geq d_0(k)$

Other angles: ∀ fixed $\alpha \in (0,1)$, setting $\lambda = (1-\alpha)/(2\alpha)$

spectral radius order $k = k(\lambda)$

= min # vertex in a graph with top eigval exactly λ

- If $k < \infty$, $N_\alpha(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor \quad \forall d \geq d_0(\alpha)$
- If $k = \infty$, $N_\alpha(d) = d + o(d)$ as $d \to \infty$

Sublinear eigenvalue multiplicity of bounded degree graphs.

A connected n-vertex graph with maximum degree Δ has second largest eigenvalue with multiplicity $\leq C \log\Delta \frac{n}{\log\log n}$